Asymptotic expansion of solution of the torsion problem for an elastic rod with a cavity and a thin bonded multilayer

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The first term of the asymptotic expansion of the solution of the torsion problem for an elastic rod is derived using the method of a matched asymptotic expansion. The prismatic rod is weakened by an internal cavity with angular points, one of which is situated on the exterior boundary. The exterior boundary of the rod is reinforced by a thin elastic multilayer. Difference between the exact and approximate solution of the problem are estimated by the norm of the Sobolev spaces. Relations for stress intensity factors in the angular points are found and verified.

1. Introduction

Structural elements reinforced by thin surface layers have found wide application in modern technology. Such elements can seriously change the elastic and strength properties of the structures. The corresponding boundary value problems have been investigated in [2, 3, 4, 20]. In those problems it is assumed that curvature of the thin layers is small. In this way, note paper [9], in which “averaged” boundary conditions are obtained for a thin surface layer with arbitrary curvature by the operator method. All the mentioned problems are related to the so-called boundary value problems with regular perturbations of the boundaries [7, 8].

However, in the cases when stress concentrators are situated near the thin layer, singular perturbations of the boundaries appear. The methods of solution of such problems have been proposed in [6, 12, 19]. One of them is the method of matched asymptotic expansion. It consists in the solution of the limiting (internal and external) problems, and later – in their coordination in some intermediate region [6, 12].

In paper [15] the method of solving the boundary value problems in infinite domains represented by wedges and layers is proposed. For some values of the parameters, homogeneous problems discussed in [15] have nontrivial solutions, which are of some class of solutions of the internal limiting boundary value problems. These solutions can be calculated by functions belonging to the kernel of special singular integral operators [14, 15]. In [13] the numerical method of deriving the functions from the kernel of the operators has been introduced.

In the paper, a singular perturbed boundary value problem is considered, which corresponds to the torsion problem of a prismatic rod with a cavity and a thin multilayer. A similar problem for a homogeneous rod with a linear crack was investigated in [1].

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2. Formulation of the problem

Let us consider a domain $\Omega_h$ with compact closure $\overline{\Omega}_h \subset \mathbb{R}^2$, smooth exterior boundary $\Gamma_e$ (for example, $\Gamma_e \in C^1$), and piecewise smooth interior boundary $\Gamma_0$ ($\partial \Omega_h = \Gamma_e \cup \Gamma_0$). By $\Gamma_1$ we denote the closed curve: $\Gamma_1 = \{ P \in \Omega_h : \text{dist}(P, \partial \Omega_h) = h \}$, (see Fig. 1).

Assume that $A, B \in \Gamma_0$ are corner points which divide the closed curve $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$, and

\begin{align}
(2.1) \quad & \text{(i)} \quad \text{dist}(A, \Gamma_e) = h \ll 1, \quad r_{\Omega_e} \geq 1, \quad \text{dist}(B, \Gamma_e) \sim 1, \\
& \text{(ii)} \quad \angle(\Gamma_1, \Gamma_0^+)_{|A} = \pi/2 \mp \phi_A, \quad \angle(\Gamma_0^+, \Gamma_0^-)_{|B} = 2\phi_B, \\
& \text{(iii)} \quad k_{\Gamma_1}(A) = k_{\Gamma_0^+}(A) = k_{\Gamma_0^-}(B) = 0,
\end{align}

where $\phi_A, \phi_B \in (0, \pi/2)$, $k_{\Gamma_1}(A)$, $k_{\Gamma_0^+}(A)$ are curvatures of the curves $\Gamma_1$, and $\Gamma_0^+$ in point $A$, but $r_{\Omega_e} = \sup\{r : B_r \subset \Omega_e\}$ is the Chebyshev radius of the domain $\Omega_e$ (here $\partial \Omega_e = \Gamma_e$, and $B_r$ is open disk of a radius $r$).

Let $(s, n)$ be a local coordinate system connected with the curve $\Gamma_1$. Its origin is at the point $A \in \Gamma_1$, and $n > 0$ along the outer normal. A Cartesian coordinate system $(x, y)$ coincides with the local system $(s, n)$ at point $A$ ($A = (0, 0)$).

If $m \in \mathbb{N}$, $\mu_0, \mu_j \in \mathbb{R}^+$ ($j = 1, 2, \ldots, m$) are some positive constants, and $0 = h_0 < h_1 < \ldots < h_j < \ldots < h_{m-1} < h_m = h$, then we consider the step function:

\begin{equation}
(2.2) \quad \mu(s, n) = \begin{cases} 
\mu_{j+1}, & (s, n) \in \Omega_h \wedge h_{j} < n < h_{j+1} , \\
\mu_0, & (s, n) \in \Omega_h \wedge -\infty < n < 0,
\end{cases}
\end{equation}

and from the assumption it follows

\begin{equation}
(2.3) \quad 0 < \min_{0 \leq j \leq m} \{ \mu_j \} = \mu \leq \mu(x, y) \leq \overline{\mu} = \max_{0 \leq j \leq m} \{ \mu_j \} < \infty.
\end{equation}
We shall use also the symbols \((j = 0, 1, \ldots, m)\):

\[
\Omega^j_h = \Omega_h \cap \{(x, y) : \mu(x, y) = \mu_j\}, \\
\Gamma^j_{j+1} = \{(s, n) : (s, n) \in \Omega_h \wedge n = h_j\}.
\]

We shall seek a harmonic function \(u(x, y)\) in each domain \(\Omega^j_h\) (the torsion function \([18]\)), satisfying along the interior boundaries \(\Gamma^j_j (j = 1, 2, \ldots, m)\) between different materials the conditions:

\[
(u_j - u_{j-1})|_{\Gamma_j} = 0, \quad \frac{\partial}{\partial n}(\mu_j u_j - \mu_{j-1} u_{j-1})|_{\Gamma_j} = f_j(x, y).
\]

But along \(\partial \Omega_h\) we have

\[
\mu m \frac{\partial}{\partial n} u_m|_{\Gamma_e} = f_m(x, y), \quad \mu_0 \frac{\partial}{\partial n} u_0|_{\Gamma_0^\pm} = -f_0^\pm(x, y),
\]

with some functions \(f_j, f_0^\pm \in C^\infty(\Gamma_j)\) (see \([18]\)), so that the following conditions are satisfied:

\[
f_j(0, h_j), f_0^\pm(0, 0) = 0, \quad \frac{\partial}{\partial s} f_j(0, h_j), \frac{\partial}{\partial s} f_0(0, 0) \sim 1.
\]

For solvability of the problem we should assume, in addition \([18]\), that

\[
\sum_{j=0}^{m+1} \int_{\Gamma_j} f_j(s) \, ds = 0,
\]

where \(\Gamma_{m+1} = \Gamma_e\), but to secure the uniqueness of the solution we normalize it by the condition:

\[
u(B) = 0.
\]

Using the results from \([10]\), one can show that the linear problem \((2.4) - (2.8)\) has the unique solution \(u_h\) in the space \(W^1_2(\Omega_h, B) = \{u \in W^1_2(\Omega_h) \wedge u(B) = 0\}\). It can be easily seen on the basis of the results of \([5]\), that the solution belongs to \(C^\infty(\overline{\Omega_h})\). Besides, we can prove that \(u_h \in C(\overline{\Omega_h})\), however, \(u_h \notin W^2_2(\Omega_h)\). To verify the first fact, it is sufficient to investigate the asymptotic behaviour of the solution near any point situated on the interior boundary \(\Gamma_j (j = 1, \ldots, m)\); but to check the second conclusion, we should know the behaviour of the solution in the neighbourhood of points \(A\) or \(B\). We shall consider in detail only the second proposition. Namely, let us represent the solution near these points in the form: \(u_h = \chi(r/\varepsilon) u_h + (1 - \chi(r/\varepsilon)) u_h\) with some small \(\varepsilon > 0\) \((\varepsilon < h_1)\). Here and further on, by \(\chi \in C^\infty(\mathbb{R}_+)\), we shall understand a cut-off function defined by

\[
\chi(t) = \begin{cases} 
1, & 0 \leq t \leq 1/3, \\
0, & 2/3 \leq t < \infty.
\end{cases}
\]
Let us note that the function \( u_\varepsilon \) is an arbitrary radius with origin at point \( A \) so that \( l \cap \Omega_h \neq \emptyset \). Then applying the Mellin transform technique to the corresponding problem for the function \( u_\varepsilon = \chi(r/\varepsilon)u_h \), and taking into account the assumptions on curvatures (2.1), we obtain

\[
\begin{align*}
u_h(h, r, \phi) &= d_A + c_A \nu_A^{-1} r^{\nu_A} F(\phi) + O(r^{\delta_A}), \quad r \to 0, \\
u_h(h, r, \phi) &= d_B + c_B \nu_B^{-1} r^{\nu_B} F(\phi) + O(r^{\delta_B}), \quad r \to 0,
\end{align*}
\]  

where \((r, \phi)\) are local coordinates connected with point \( A \) (or \( B \)), and the angle \( \phi \) calculated with respect to the bisector of the corresponding corner angles, are situated in the domains \( \Omega_h^1 (\Omega_h^0, \text{respectively}) \), but

\[
F(\phi) = \begin{cases} 
\frac{\sin \phi \nu}{\sin(\pi \nu/2)}, & |\phi| \leq \pi/2, \\
\frac{\sin \phi \cos(\pi - \phi_0 - |\phi|) \nu}{\sin(\pi/2 - \phi_0) \nu}, & \pi/2 < |\phi| < \pi - \phi_0,
\end{cases}
\]  

and \( \phi_0 = \phi_A(\phi_B), \) \( d_B = 0 \) \( (u_h \in W_2^1(\Omega_h, B)) \), but constants \( \nu_A, \nu_B \in (0, 1) \) are the first zeros of the function:

\[
\Delta_c(s) = \kappa \cos \phi_0 s - \cos(\pi - \phi_0) s,
\]  

\( \kappa_A = \frac{\mu_0 - \mu_1}{\mu_0 + \mu_1}, \quad \kappa_B = 0, \) which are the nearest to the imaginary axis. Since \( \kappa_B = 0 \), the relation for the function \( F(\phi) \) at point \( B \) has a similar form for \(|\phi| \leq \pi/2 \) as well as for \(|\phi| > \pi/2 \). Here the values of the parameters \( \delta_A, \delta_B \in (1, 2) \) in (2.11) are calculated as follows:

\[
\delta_A = \min\{\nu_A^{(2)}, \tau_A\}, \quad \delta_B = \min\{\nu_B^{(2)}, \tau_B\},
\]  

where \( \nu_A^{(2)}, \nu_B^{(2)} \) are the second zeros of the function \( \Delta_c(s) \), but \( \tau_A, \tau_B \) are the first zeros \( (\tau_A, \tau_B > 0) \) of the function: \( \Delta_s(s) = s^{-1}[\kappa \sin \phi_0 s + \sin(\pi - \phi_0) s] \), with the respective value of the parameter \( \kappa \) \( (\kappa_A, \kappa_B) \).

The constants \( c_A, c_B \) in (2.11) play an important role in fracture mechanics [17] (stress intensity factors). The next mechanical parameter which can be calculated from the solution \( u_h \) of the problem (2.5)–(2.9) is the stiffness [18]:

\[
C = \iint_{\Omega_h} \mu(x, y) \left( x^2 + y^2 + \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) u_h(h, x, y) \right) \, d\Omega.
\]  

However, the numerical process used for solving the problem (2.5)–(2.9) is difficult in view of the existence of the small parameter \( h \), and of the singularity of the solution in the neighbourhood of point \( A \) situated near the exterior boundary of the domain. Further on, we find the first term of the asymptotic expansion of the solution \( \tilde{u}_h \), which is close to \( u_h \) in the norm \( W_2^1(\Omega_h) \), and makes it possible to obtain the values of coefficients \( c_A, c_B, C \) from (2.11), (2.13).
3. Limiting boundary value problems

3.1. External problem

Now we consider similar problem but the domain will be somewhat different. Namely, by \( \Omega_0 \) we denote the simply connected domain with boundary \( \partial \Omega_0 = \partial \Omega_h \cup M_0^+ \cup M_0^- \), where \( M_0^\pm = \{(x, y) : 0 < y < h \land x = 0^\pm \} \). Along the curves \( M_0^\pm \) we define functions \( f_{M}^\pm(s) \equiv 0 \), hence, the condition (2.9) holds true and the function along the boundary \( \partial \Omega_0 \) is continuous. Problem (2.5)–(2.9) in the domain \( \Omega_0 \) also has a unique solution \( u_0 \), belonging to \( W_2^1(\Omega_0, B) \). Besides, \( u_0 \in C^\infty(\Omega_0^0), u_0 \in C(\Omega_0), \) but \( u_0 \notin C(\partial \Omega_0) \). This is because the domain \( \Omega_0 \) has not the “segment” property (see [10]), and \( u_0 \in W_2^1(\Omega_0, B) \) is a multifunction near the parts \( M_0^\pm \) of the boundary \( \partial \Omega_0 \) (as \( (x, y) \) tends to a point \( (0, y_*) \) on the boundaries \( M_0^\pm \) from different sides of the domain \( \Omega_0 \), the function \( u_0 \) has different limiting values).

The solution \( u_0 \) exhibits the asymptotic behaviour (2.11) near point \( B \) with a constant \( c_0^B \), but in the neighbourhood of the point \( A \)

\[
(3.1) \quad u_0(h, x, y) = \pm d_0^\pm + O(r^{-A}), \quad r \to 0, \quad 0 < \pm \phi < \pi - \phi_A.
\]

Hence, \( u_0 \) cannot be considered as an approximation of \( u_h \) near the zero point.

3.2. Green's function

We shall also need the Green function \( G_A(x, y) \) for this problem in the domain \( \Omega_0 \), with delta-function concentrated near point \( A \). It will be normalized by the relation (2.9). Asymptotic behaviour of the Green function near point \( B \) is of the form (2.11) (similar to \( u_h \) and \( u_0 \)) with \( d_B = 0 \) and the constant \( c = g_B \), but near the zero point

\[
(3.2) \quad G_A(h, x, y) = \pm \ln r \pm g_0^\pm + O(r^{-A}), \quad r \to 0, \quad 0 < \pm \phi < \pi - \phi_A,
\]

where \( g_0^\pm \) are some constants.

Let us note that the Green function \( G_A \) is uniquely determined, and can be calculated using the representation

\[
G_A = \chi(r/h) \cdot \text{sign} \phi \cdot \ln r + v_0,
\]

where the function \( v_0 \in W_2^1(\Omega_0, B) \) satisfies Poisson equation with the right-hand side: \( \text{sign} \phi \cdot (\ln r \Delta \chi(r/h) + 2(r/h)^{-1} \chi'(r/h)) \) and the boundary conditions (2.5), (2.6) with functions \( \hat{f}_j(s) = \frac{\partial}{\partial n} [\chi(r/h) \ln r] \) along the curves \( \Gamma_j \). All these functions are smooth, and \( \hat{f}_{M}^\pm(y) \equiv 0, \hat{f}_0^\pm(A), \hat{f}_1(A) = 0, \) in view of the assumption (2.1) for curvatures of the curves near point \( A \). Hence, the problem for the function \( v_0 \in W_2^1(\Omega_0, B) \) and the problem of the Subs. 3.1 for the function \( u_0 \in W_2^1(\Omega_0, B) \) are similar from the point of view of their numerical realization.
3.3. Internal problem

Now let us consider the infinite domain $G = G^\pm_0 \cup G_j$ represented in Fig. 2, and try to find nontrivial harmonic function $w(x', y')$ satisfying the homogeneous internal boundary conditions (2.6) along the boundaries $\zeta_{j+1} = \{(x', y') : y' = y_j = h_j/h, x' \in \mathbb{R}\}$ between the domains $G_{j-1}, G_j$ ($j = 1..., m$), and homogeneous conditions (2.8) along the boundaries $\zeta_{m+1}, \zeta_0^-$.

![Fig. 2.](http://rcin.org.pl)

At infinity we assume, in addition, that $w = O(\ln r), r \to \infty$. There are two linearly independent harmonic functions satisfying such conditions: $w_1(x', y') =$ const – even function with respect to argument $x'$, and odd function $w_2(x', y')$. The function $w_2(x', y')$ can be calculated, using the inverse Fourier transform, by the nontrivial solution $z(\xi)$ of the singular integral equation obtained in [15] (the corresponding equation (3.16)). From theorem B.4 [15], it follows that $z \in W^{p,\alpha,\beta}_l(\mathbb{R}^+)$ for any $l \in \mathbb{N}, p \in [1, \infty), \alpha > 0, \beta < \nu_A$, and

$$
\begin{align*}
z(\xi) &= \ln \xi + z_0 + O(\xi^2), \quad \xi \to 0, \\
z(\xi) &= z_\infty \xi^{-\nu_A} + O(\xi^{-\nu_A(2)}), \quad \xi \to \infty.
\end{align*}
$$

Here, $W^{p,\alpha,\beta}_l(\mathbb{R}^+)$ is the space of functions, which are summable (together with their $l$-derivatives) with a special weight (see [14]). The space $W^{p,\alpha,\beta}_l(\mathbb{R}^+)$ does not coincide with usual Sobolev spaces $W^l_p(\mathbb{R}^+)$. In turn, the method of numerical calculation of this nontrivial solution has been proposed in [13]. Finally, $w_2(x', y')$ can be determined (with accuracy to a multiplier) from the relation:

$$
\begin{align*}
w_2(x', y') &= 2 \int_0^\infty \left[ \text{ch} y'\xi + \left[ \xi \mu_1 M_p(\xi) \right]^{-1} \text{sh} y'\xi \right] z(\xi) \sin(x' \xi) d\xi, \\
(x', y') &\in G_1,
\end{align*}
$$
(3.4) \[ w_2(x', y') = \frac{1}{\pi i} \int_{-i\infty - \delta}^{i\infty - \delta} \Gamma(s) \sin(\pi s/2) \frac{\cos(\pi - \phi_A - \phi)s}{\cos(\pi /2 - \phi_0)s} \cdot \int_0^\infty z(\xi)(r\xi)^{-s} d\xi ds, \quad (x', y') \in G_0^+, \]

where \(0 < \delta < \nu_A\), the function \(M_p(\xi)\) can be calculated by recurrence formulae from [15], and besides, \(M_p(\xi) = O(\xi^{-2}), \xi \to 0, M_p(\xi) = -(\mu_1 \xi)^{-1} + O(e^{-2\xi x_1}), \xi \to \infty.\)

Using this information, we can show that the asymptotic behaviour of the function \(w_2(x', y')\) near the zero point is of the form (2.11), with the constant \(c_w = 2\pi^{-1}z_\infty \Gamma'(1 - \nu_A)\sin(\pi \nu_A/2), d_w = 0,\) and \(\nu_A^{(2)}\) instead of the parameter \(\delta;\) but at infinity we obtain

(3.5) \[ w_2(x', y') = \pm \begin{cases} \ln r + \gamma + z_0, & (x', y') \in G_0, \\ \ln |x'| + \gamma + z_0, & (x', y') \in G_j, \end{cases} + O(r^{-2}), \quad r \to \infty, \quad \pm x' > 0, \]

where \(\gamma = \Gamma'(1)\) is the Euler constant.

4. Main result

Using the method of matched asymptotic expansion (see [6, 19]), we shall consider function \(w_2(s/h, n/h) + \text{const}\) as an approximation of the solution \(u_h\) in the neighbourhood of point \(A,\) but a linear combination of the functions \(u_0(h, x, y), G_A(h, x, y)\) in the remaining part of domain \(\Omega_h.\) Let \(\alpha \in (0, 1)\) be some constant, and

(4.1) \[ \tilde{u}_h^{(1)}(h, x, y) = (1 - \chi(r/h^\alpha))[u_0(h, x, y) + DG_A(h, x, y)] + \chi(r/h^\alpha)[Dw_2(s/h, n/h) + E]. \]

Unknown constants \(D, E\) should be calculated in such a way that both parts (internal and external) of the solution (4.1) will coincide on the distance \(r = h^\alpha/2:\)

(4.2) \[ u_0(h, x, y) + DG_A(h, x, y) - Dw_2(s/h, n/h) - E = O\left(h_{\min}\{\tau A, 2 - 2\alpha\}\right), \]

\[ \nabla[u_0(h, x, y) + DG_A(h, x, y) - Dw_2(s/h, n/h) - E] = O\left(h_{\min}\{\tau A, 2 - 2\alpha\} - \alpha\right), \]

for \(h^\alpha/3 < r < 2h^\alpha/3\) uniformly with respect to the angular coordinate \(\theta;\) then

(4.3) \[ D = \frac{d_0^+ + d_0^-}{2(\gamma + \ln h) - h_0^+ - h_0^-}, \quad E = \frac{1}{2}[d_0^+ - d_0^- + D(g_0^+ - g_0^-)]. \]
Let us note, that the function $\tilde{u}^{(1)}_h$ from (4.1) belongs to the space $W^1_2(\Omega_h, B)$, and the constants in the main terms of asymptotics (2.11) near points $A, B$ are:

\begin{equation}
\tilde{\epsilon}_A = \frac{2}{\pi} z_{\infty} D h^{-\nu_A} \Gamma(1 - \nu_A) \sin(\pi \nu_A/2), \quad \tilde{\epsilon}_B = \epsilon^0_B + D g_B.
\end{equation}

**Theorem 1.** Let $\alpha \in (0, 1)$ and $h \ll 1$, then for the function $\tilde{u}^{(1)}_h \in W^1_2(\Omega_h, B)$, the following estimates hold true:

\[
\|u_h - \tilde{u}^{(1)}_h\|_{W^1_2} = O\left(h^{\min\{\alpha(\tau_A - 1), 2 - 3\alpha\}}\right),
\]

\[
C - \tilde{C} = O\left(h^{\min\{\alpha(\tau_A - 1), 2 - 3\alpha\}}\right),
\]

\[
c_A - \tilde{c}_A = O\left(h^{\min\{\alpha(\tau_A - \nu_A), 2 - \alpha(2 + \nu_A)\}}\right),
\]

\[
c_B - \tilde{c}_B = O\left(h^{\min\{\alpha(\tau_A + \nu_A), 2 - \alpha(2 - \nu_A)\}}\right).
\]

**Proof.** First of all note, that the difference between $u_h$ and $\tilde{u}^{(1)}_h$ in each domain $\Omega^j_h$ satisfies the Poisson equation with the right-hand side $R^{(1)}(h, x, y)$:

\[
R^{(1)}(h, x, y) = R^{(1)}_1(h, x, y) - R^{(1)}_2(h, x, y),
\]

\[
R^{(1)}_1(h, x, y) = [u_0(h, x, y) + DG_A(h, x, y) - Dw_2(s/h, n/h) - E]\Delta \chi(r/h^\alpha) + 2[\nabla u_0(h, x, y) + DG_A(h, x, y) - Dw_2(s/h, n/h) - E]\nabla \chi(r/h^\alpha),
\]

\[
R^{(1)}_2(h, x, y) = D\chi(r/h^\alpha)\Delta x y w_2(s/h, n/h),
\]

and fulfills the boundary conditions (2.5), (2.6) with the functions

\[
\tilde{f}^{(1)}_j = \chi(r/h^\alpha)f_j + (\mu_{j-1} - \mu_j)[u_0 + DG_A - Dw_2(s/h, n/h)] - E\frac{\partial}{\partial n}\chi(r/h^\alpha),
\]

\[
\tilde{f}^{(1)}_0 = \chi(r/h^\alpha)f_0 - \mu_0[u_0(x, y) + DG_A(x, y) - Dw_2(s/h, n/h)] - E\frac{\partial}{\partial n}\chi(r/h^\alpha),
\]

\[
\tilde{f}^{(1)}_m = \chi(r/h^\alpha)f_m + \mu_m[u_0(x, y) + DG_A(x, y) - Dw_2(s/h, n/h)] - E\frac{\partial}{\partial n}\chi(r/h^\alpha),
\]

instead of $f_j$. Such a problem (for the function $u_h - \tilde{u}^{(1)}_h$) has also a unique solution in the space $W^1_2(\Omega_h, B)$. 
Taking into account (4.2), we can obtain for $h \to 0$
\[
R_1^{(1)}(h, x, y) = O(h^{\min\{\alpha(\tau_A-2), 2-4\alpha\}}),
\]
\[
\text{supp} R_1^{(1)} = \{(x, y) \in \Omega_h : h^\alpha / 3 < r < 2h^\alpha / 3\},
\]
but to estimate the function $R_2^{(1)}(h, x, y)$ ($\text{supp} R_2^{(1)} = \{(x, y) \in \Omega_h : 0 < r < h^\alpha / 3\}$), the Laplace operator should be considered in the curvilinear coordinate system $(s, n)$:
\[
\Delta_{x,y} w_2(s/h, n/h) = \frac{1}{1- nk(s)} \left[ \frac{\partial}{\partial n} \left((1- nk(s)) \frac{\partial w_2}{\partial n}\right) + \frac{\partial}{\partial s} \left(\frac{1}{1- nk(s)} \frac{\partial w_2}{\partial s}\right) \right].
\]

Denoting $\xi = s/h, \eta = n/h$, we can conclude, in view of assumption (2.1) on the curves $\Gamma_1$, and taking into account the asymptotic formula (3.5) for the function $w_2$, that $R_2^{(1)}(h, x, y) = R_3^{(1)}(\xi, \eta) + O(h)$, where
\[
R_3^{(1)}(\xi, \eta) = O(\rho^\nu), \quad \rho \to 0, \quad R_3^{(1)}(\xi, \eta) = O(1), \quad \rho \to \infty.
\]

The functions $\tilde{f}^{(1)}_j$ in the boundary conditions (2.5), (2.6) can be represented as a sum $\tilde{f}^{(1)}_j = \tilde{f}^{(1)}_{j1} + \tilde{f}^{(1)}_{j2}$, which at $h \to 0$ have the properties:
\[
\tilde{f}^{(1)}_{j1} = O(h^\alpha), \quad \text{supp} \tilde{f}^{(1)}_{j1} = \{(x, y) \in \Omega_h : 0 < r < 2h^\alpha / 3\},
\]
\[
\tilde{f}^{(1)}_{j2} = O(h^{\min\{2-3\alpha, \alpha(\tau_A-1)\}}), \quad \text{supp} \tilde{f}^{(1)}_{j2} = \{(x, y) \in \Omega_h : h^\alpha / 3 < r < 2h^\alpha / 3\}.
\]

We can then conclude that
\[
\| R_1^{(1)} \|_{L_2(\Omega_h)} = O(h^{\min\{\alpha(\tau_A-1), 2-3\alpha\}}), \quad \| R_2^{(1)} \|_{L_2(\Omega_h)} = O(h^\alpha),
\]
\[
\| \tilde{f}^{(1)}_{j1} \|_{L_2(\Gamma_j)} = O(h^{\min\{\alpha(\tau_A-1/2), 2-5\alpha / 2\}}), \quad \| \tilde{f}^{(1)}_{j2} \|_{L_2(\Gamma_j)} = O(h^{3\alpha / 2}).
\]

Now, the first conclusion of Theorem 1 follows from the results [10]. However, the constant in the estimate ($\| u_h - \tilde{u}_h^{(1)} \| \leq \text{Const} h^{\min\{\alpha(\tau_A-1), 2-3\alpha\}}$) cannot be effectively obtained. It depends on the norm of the inverse operator of problem (2.5)–(2.9). The second relation follows immediately from the H"older inequality.

For estimation of the constants $c_A, c_B$ in the main terms of the asymptotics (2.11), we shall use the MAZ'YA, PLAMENEVSKY method [11]. Following [11] (see also [17]), we can define “non-energetic” harmonic function $\Psi_A^- \in L_2(\Omega_h)$ satisfying the homogeneous problem (2.5)–(2.9) with asymptotic behaviour (2.11) near point $B$, but in the neighbourhood of point $A$ satisfying the condition
\[
\Psi_A^-(x, y) = r^{-\nu A} F(\phi) + O(1), \quad r \to 0,
\]
where function $F(\phi)$ is defined in (2.11). The function $\Psi^-_{x}(x, y)$ can be calculated from the representation ($\varepsilon < h_1$):

$$\Psi^-_{x}(x, y) = \chi(r/\varepsilon) r^{-\nu_A} F(\phi) + \Psi^-_\varepsilon(x, y), \quad \Psi^-_\varepsilon \in W^1_2(\Omega_h, B),$$

because the corresponding problem for function $\Psi^-_\varepsilon$ has a unique solution in the space $W^1_2(\Omega_h, B)$. Further on we define $\omega_\varepsilon = \{(x, y) : r < \varepsilon\}$ and write the Green formulae for the functions as $\hat{u}_h = u_h - \hat{u}_h^{(1)}$ and $\Psi^-_{x}$ in the domains of $\Omega^0_h \setminus \omega_\varepsilon$, $\Omega^1_h \setminus \omega_\varepsilon$, $\Omega^j_h (j = 2, \ldots, m)$. The sum of the corresponding relations is in the form of:

$$\int_{\Omega^m_h \setminus \omega_\varepsilon} \mu(x, y) \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] d\Omega = \int_{\Gamma_{m+1}} \mu_m \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] d\sigma$$

$$+ \sum_{j=1}^{m} \int_{\Gamma_j \cap (\Omega_h \setminus \omega_\varepsilon)} \left\{ \mu_j \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] - \mu_{j-1} \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] \right\} d\sigma$$

$$- \int_{\Gamma_0 \cap (\Omega_h \setminus \omega_\varepsilon)} \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] d\sigma - \int_{\partial \omega_\varepsilon} \mu(x, y) \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] d\sigma,$$

or taking into account the equations and the boundary conditions for functions $\hat{u}_h$ and $\Psi^-_{x}$, this relation can be rewritten as follows ($\varepsilon < h_1$):

$$\int_{\partial \omega_\varepsilon} \mu(x, y) \left[ \frac{\partial \hat{u}_h}{\partial n} - \hat{u}_h \frac{\partial \Psi^-_{x}}{\partial n} \right] d\sigma$$

$$= \sum_{j=2}^{m+1} \int_{\Gamma_j} \Psi^-_{x} f_{j}^{(1)} d\sigma + \int_{\Gamma_1 \cap (\Omega_h \setminus \omega_\varepsilon)} \Psi^-_{x} f_{1}^{(1)} d\sigma + \int_{\Gamma_0 \cap (\Omega_h \setminus \omega_\varepsilon)} \Psi^-_{x} f_{0}^{(1)} d\sigma$$

$$- \int_{\Omega^m_h \setminus \omega_\varepsilon} \mu(x, y) \Psi^-_{x} \left[ R^-_{1}(h, x, y) - R^-_{2}(h, x, y) \right] d\Omega.$$

The net result will be obtained by passing to the limit $\varepsilon \rightarrow 0$:

$$\Phi(\mu_0, \mu_1, \phi_A)(c_A - \bar{c}_A)$$

$$= \sum_{j=0}^{m+1} \int_{\Gamma_j} \Psi^-_{x} \left[ f_{j}^{(1)} + f_{j}^{(2)} \right] d\sigma - \int_{\Omega_h} \mu \Psi^-_{x} \left[ R^-_{1} - R^-_{2} \right] d\Omega.$$

Here we use information (2.11) and (4.5) about the asymptotic behaviour of the functions $\hat{u}_h$, $\Psi^-_{x}$ near point $A$ for calculating the integral on the left-hand side.
of (4.6):

\[
\Phi(\mu_0, \mu_1, \phi_A) = 2 \left\{ \mu_1 \frac{\pi \nu_A - \sin \pi \nu_A}{1 - \cos \pi \nu_A} + \mu_0 \frac{(\pi - 2\phi_A)\nu_A + \sin(\pi - 2\phi_A)\nu_A}{1 + \cos(\pi - 2\phi_A)\nu_A} \right\}.
\]

The first and the fourth terms on the right-hand side of (4.7) are estimated as \(O(h^{\alpha(2 - \nu_A)})\), but the remaining two terms are \(O(h^{\min\{\alpha(\tau_A - \nu_A), 2 - \alpha(2 + \nu_A)\}})\). Consequently, the third conclusion of Theorem 1 is proved. The remaining estimation of Theorem 1 is performed in a similar manner. For this purpose, we should take the “non-energetic” function \(\Psi^-_B\) (instead of \(\Psi^-_A\)), which exhibits the asymptotic behaviour (2.11) near the point \(A\), but in the neighbourhood of point \(B\) in the form of (4.5) with \(\nu_B\). Then, repeating the same reasoning, we obtain the fourth conclusion of Theorem 1. Let us note that the constants in the last two estimates have been obtained effectively.

**Corollary 1.** The optimal value of the parameter \(\alpha\) is \(\alpha_* = 2/(2 + \tau_A)\), then the estimates are:

\[
\|u_h - \tilde{u}_h^{(1)}\|_{W_2^1}, |C - \tilde{C}| = O(h^{2 - 3\alpha_*}),
\]

\[
c_A - \tilde{c}_A = O(h^{2 - \alpha_*(\tau_A + \nu_A)}), \quad c_B - \tilde{c}_B = O(h^{2 - \alpha_*(\tau_A - \nu_A)}).
\]

**Remark 1.** As it follows from the proof of Theorem 1, the results would be improved, if we could more precisely estimate the terms of solution \(u_0\) and the Green function \(G_A\) of the asymptotic behaviour: \(O(r^{\tau_A}), r \to 0\). For this purpose, note that the corresponding problem for function \(u_0\) is the perturbation boundary value problem with the regular boundary layer near \(\Gamma_{m+1} = \Gamma_e\). The main terms of such problems have been constructed in [4]. Basing on the results from [4], one can show that the term \(O(r^{\tau_A})\) in (3.1) can be estimated as: \(\text{const}(h)F_\star(\phi)r^{\tau_A}\), where \(\text{const}(h) = O(h^\beta)\) with some \(0 < \beta \leq \tau_A - \tau_A\). Here, \(\tau_A^0\) is the corresponding parameter in (3.1) for the solution \(u_0^0(x, y)\) of the nonperturbed problem \((\mu(x, y) = \mu_0, h = 0)\). In a similar manner, the estimation of the corresponding term of the Green function (3.2) can be obtained. Then we can formulate

**Theorem 2.** Let \(\alpha \in (0, 1)\) and \(h \ll 1\), then for function \(\tilde{u}_h^{(1)} \in W_2^1(\Omega_h, B)\) estimates hold true:

\[
\|u_h - \tilde{u}_h^{(1)}\|_{W_2^1} = O(h^{\min\{\alpha, \beta + \alpha(\tau_A - 1), 2 - 3\alpha\}}),
\]

\[
|C - \tilde{C}| = O(h^{\min\{\alpha, \beta + \alpha(\tau_A - 1), 2 - 3\alpha\}}),
\]

\[
c_A - \tilde{c}_A = O(h^{\min\{\alpha, \beta + \alpha(\tau_A - \nu_A), 2 - \alpha(2 + \nu_A)\}}),
\]

\[
c_B - \tilde{c}_B = O(h^{\min\{\alpha, \beta + \alpha(\tau_A + \nu_A), 2 - \alpha(2 - \nu_A)\}}).
\]
Corollary 2. Then the optimal value of the parameter \( \alpha \) in Corollary 1 is

\[
\alpha_* = \max \left\{ \frac{1}{2}, \frac{2-\beta}{2+T_A} \right\}.
\]

5. Remarks and conclusions

In this section we propose some generalizations under which the mentioned results of the theorems will hold true.

First of all note that from \([18]\) it follows that

\[
\begin{align*}
f_j &= (\mu_j - 1) [y \cos(n, x) - x \cos(n, y)], \\
f_m &= \mu_m [y \cos(n, x) - x \cos(n, y)], \\
f_0 &= \mu_0 [y \cos(n, x) - x \cos(n, y)].
\end{align*}
\]

Consequently, these functions satisfy the conditions (2.7). Nevertheless, the results still remain valid, if the functions are “little affected” in the neighbourhood of point \( A \).

For this purpose, it is sufficient to find the solution in the form:

\[
u_h = \tilde{u}_h + \chi(r/h)v_1(s, n),
\]

where the function is

\[
v_1 = a_j + b_j s + c_j n \quad \text{in each region } \Omega^1_h.
\]

The constants \( a_j, b_j, c_j \) should be calculated so that \( v_1 \) is continuous along \( \Gamma_j \), but for function \( \tilde{u}_h \) the conditions (2.7) have been satisfied.

Further on, note that the conditions (iii) in (2.1) can be weakened like this:

\[
k_{l^1_1}(A), k_{l^1_0}(A), k_{l^1_0}(B) \sim 1.
\]

The angle of corner \( A \) can be nonsymmetric with respect to the normal to the boundary \( \Gamma_0 \) at this point, in contrast to (ii). Then the functions \( F(\phi) \) in (2.11) and the transcendental functions \( \Delta_s(s) \) (necessary to determine the parameters \( \tau, \nu \)) should be corrected; but the corresponding internal boundary value problems can be calculated by solving of the systems of singular integral equations [15], instead of the singular integral equations as it is in the symmetric cases.

The step function \( \mu(x, y) \) allows for the following generalization:

1. The boundaries of discontinuity \( \Gamma_j \) of function \( \mu(x, y) \) can be defined as in (2.2) with functions \( h_j(s) \) instead of parameters \( h_j \). We should assume only that:

\[
h_j(s) > h_{j-1}(s), \quad h_m(s) = O(h), \quad h'_j(0) = 0, \quad h''_j(0) \sim 1.
\]

2. In each domain \( \Omega^j_h \) the conditions are true:

\[
\begin{align*}
\mu \in C^2(\Omega^j_h), \quad \frac{\partial}{\partial x}\mu(0, y) &= 0, \\
\frac{\partial^2}{\partial x^2}\mu(0, y) &\sim 1, \quad (0, y) \in \Omega^j_h (j > 0), \\
\frac{\partial}{\partial r}\mu(0, \theta) &= 0, \quad \frac{\partial^2}{\partial r^2}\mu(0, \theta) \sim 1, \quad (0, \theta) \in \Omega^0_h.
\end{align*}
\]

The function \( \mu(x, y) \) depends weakly on the argument \( x \) in the multilayer near the angle vertex. Then we shall find solution \( u_h \) of equation \( \nabla(\mu(x, y)\nabla u_h) = 0 \) instead of the Laplace equation \( \Delta u_h = 0 \) used in the paper. Such a problem corresponds to the general case of a nonhomogeneous elastic rod. Note in this connection that the internal boundary value problems (Sec. (3.2)) can be also solved in this case by the method [15] (see Appendix in [16]).

The boundary conditions can be also generalized. Namely, the first of the conditions (2.5) can be represented in the form:

\[
[u_h] - a(s) \frac{\partial}{\partial n} u_h = \tilde{f}_j, \quad a'(0) = 0, \quad a''(0) \sim 1, \quad \text{instead of } [u_h] = 0.
\]

The corresponding internal boundary value problems can be solved by the same method [15, 16].
Let us note in conclusion, that the first two conditions (i) cannot be modified, of course (these conditions make it possible to use the asymptotic methods). If the third condition is not true and \( \text{dist}(B, I_c) = O(h) \), then the asymptotic expansion of the solution can also be constructed. However, the corresponding external boundary value problems are different from those shown in the paper (Sec. (3.1)), and the representation of the solution (4.1) should be changed. In [19], such a problem in a homogeneous domain with the linear crack has been considered.

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