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On the singular integral equations approach to the interface crack problem for piezoelectric materials

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A PLANE STRAIN PROBLEM for an interface crack with poling axis orthogonal to the crack plane is considered. The contact zone model with an artificial contact zone is considered for electrically permeable crack. By means of the method of singular integral equations, the quasi-invariance of the energy release rate with respect to the contact zone length is demonstrated. The appearance of a singularity of real power type instead of an oscillating singularity for insulated crack faces for most combinations of ceramics is shown. In a numerical way, the comparison of stresses and electrical displacements corresponding to the different interface crack models is employed.

1. Introduction

FOR COMPOSITE MATERIALS and, in particular, for piezoelectric compounds, interfacial fracture is common and determines mainly the materials overall strength properties. Due to their significance in the areas of mechanics, material science, and engineering, interfacial cracks have been the focus of great interest in the last decades.

In the fundamental papers of WILLIAMS [1], CHEREPANOV [2], ENGLANG [3], RICE and SIH [4] and ERDOGAN [5], the oscillating model for cracks between two isotropic materials has been developed. This model was re-examined afterwards in numerous papers and particularly in the paper of RICE [6]. Interfacial cracks between anisotropic materials have been studied, for instance, by TING [7].

It is well known that the oscillating crack model leads to oscillations in the stress field near the crack tip, resulting in the physically unrealistic phenomenon of interpenetrating crack faces. Therefore, COMNINOU [8] proposed a crack model making use of zones of frictionless contact. This contact zone model was investigated numerically by COMNINOU [9], DUNDURS and COMNINOU [10] and confirmed in an analytical way by Atkinson [11], Simonov [12], and Gautsen and Dundurs [13].

Concerning piezoelectric materials, it is worth to note that there are few publications in the literature studying interfacial cracks. Such materials are of high interest in modern smart technology applications. In sensors, they allow for converting mechanical state changes into electric measuring signals, while in actuators, they offer the possibility to convert an electric control signal into mechanical action. In addition to the appearance of a large number of different constants, this case is connected with the coupling of mechanical and electric fields.

A further complication concerns the formulation of the electric boundary conditions at the crack faces in piezoelectric materials. Taking into account the permittivity of the medium filling the crack, increases the complexity of the problem significantly (Dunn [14], Hao and Shen [15], Balke *et al.* [16]). Therefore, it is common use to employ the idealised boundary conditions of permeable or insulating crack faces, respectively. Out of these two extreme cases, the permeable boundary condition which simply ignores the crack electrically, seems to be the more realistic one, if results are compared to the analyses taking into account the permittivity of the crack (Gruebner and Kamlah [17]). There are also opposite opinions based on the fact that the permittivity of piezoceramics is three orders of magnitude higher than the one of air or vacuum (Suo *et al.* [18], Park and Sun [19]). For a detailed discussion of this topic see McMeeking [20].

For permeable interfacial crack faces between piezoelectric materials, an exact oscillating solution for a crack between a conductor and a piezoelectric material was obtained in the paper of Kudryavtsev *et al.* [21]. Recently, the character of the singularity at the interface crack tip of permeable and insulating crack faces has been investigated in the papers of Kuo and Barnet [22] and Suo *et al.* [18] for piezoelectric compounds. In particular, the possibility of the existence of real as well as oscillating singularities at an interface crack tip was predicted in these papers.

A closed crack tip model for an interface crack in thermopiezoelectric materials was considered by Qin and Mai [23]. Insulated crack faces were considered in this paper and stress intensity factors and the size of the contact zone were found in a numerical way for a particular piezomaterial group.

In the present paper, a plane strain problem for an interface crack between two piezoelectric materials is considered. By means of the Fourier transform, the problem is reduced to a system of singular integral equations corresponding to different electric conditions at the crack faces and different interface crack models. The numerical solution of these systems gave the possibility to find the main fracture characteristics for all the considered cases. For the case of permeable crack faces, artificial contact zones are introduced and the important property of the quasi-invariance of the energy release rate with respect to the contact zone length is proved. On the basis of this property, a numerical method for investigation of an interface crack problem for a finite-sized body is suggested. The real contact zone length is found for specific choices of the material constants. For insulating crack faces, the crack model without contact zone (open crack model) is considered. The possibility of the existence of a real as well as an oscillating singularity is revealed. Namely, the new principal result related to the most important class of piezoceramics concerning the simultaneous existence of a real non-square root singularity or an oscillating singularity and square root singularity is found out.

It is worth to note that the main advantage of the suggested approach is connected with its possibility to investigate completely different interface crack models and various conditions on the crack faces. Namely, this approach presents both a way for the determination of the singularities as well as a method for the solution of the boundary value problem.

2. Determination of the interface relations

A plane strain state of two bonded semi-infinite spaces $z < 0$ and $z > 0$ is considered. It is loaded by $\sigma_{zz}^{\pm} = \sigma_{zz}^{\infty}$, $D_z^{\pm} = D_z^{\infty}$ at infinity (the "+" sign refers to the upper domain, and "−" to the lower one). A plane crack $z = 0$, $|x| < b$ with free faces is situated in the interface (Fig. 1).

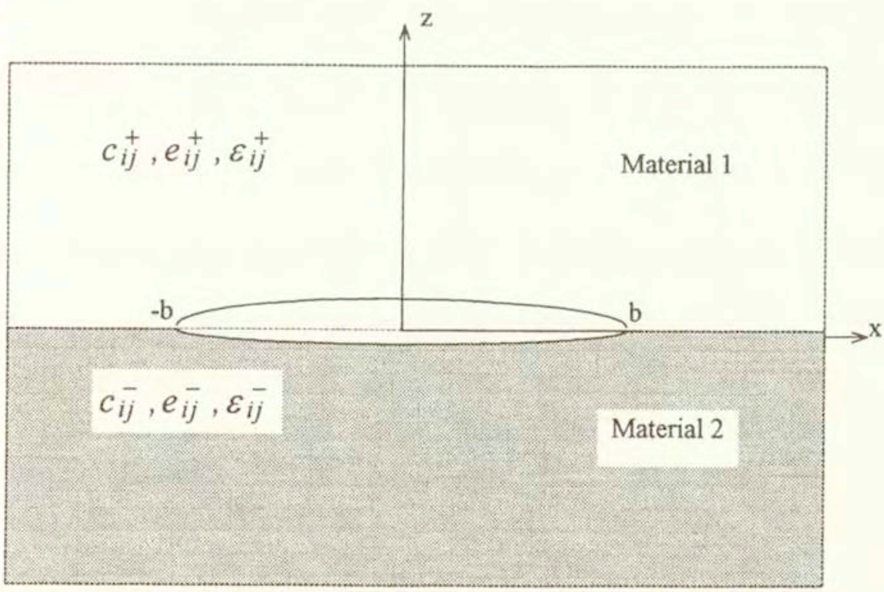

Fig. 1. An interface crack between two semi-infinite piezoelectric planes.

We assume that both materials belong to the hexagonal class of symmetry 6 mm. The plane deformation is orthogonal to the y axis and the z axis coincides with the vector of polarisation. We then find from the classical linear piezoelectric constitutive law the following equations (see e.g. Parton and Kudryavtsev [24]):

$$\begin{aligned} \sigma_{xx} &= c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \varphi}{\partial z}, \\ \sigma_{zz} &= c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \varphi}{\partial z}, \\ \sigma_{xz} &= c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \varphi}{\partial x}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} D_x &= e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \varphi}{\partial x}, \\ D_z &= e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \varphi}{\partial z}. \end{aligned} \tag{2.2}$$

Here, $\{\sigma_{xx}, \sigma_{zz}, \sigma_{xz}\}$ represent the stress components, $\{u, w\}$ are the displacement components, $\{D_x, D_z\}$ are the electric displacement components and φ is the electric potential; $\{c_{11}, c_{13}, c_{33}, c_{44}\}$ are the elastic constants, $\{e_{31}, e_{33}, e_{15}\}$ are the piezoelectric constants and $\{\varepsilon_{11}, \varepsilon_{33}\}$ are the dielectric permittivities.

Taking into account (2.1), (2.2), the equations of equilibrium and electrostatic equations yield the following equations with respect to the displacements and electric potential:

$$\begin{aligned} c_{11}\frac{\partial^2 u}{\partial x^2} + c_{44}\frac{\partial^2 u}{\partial z^2} + (c_{13}+c_{44})\frac{\partial^2 w}{\partial x \partial z} + (e_{31}+e_{15})\frac{\partial^2 \varphi}{\partial x \partial z} &= 0, \\ (c_{13}+c_{44})\frac{\partial^2 u}{\partial x \partial z} + c_{44}\frac{\partial^2 w}{\partial x^2} + c_{33}\frac{\partial^2 w}{\partial z^2} + e_{33}\frac{\partial^2 \varphi}{\partial z^2} + e_{15}\frac{\partial^2 \varphi}{\partial x^2} &= 0, \\ (e_{31}+e_{15})\frac{\partial^2 u}{\partial x \partial z} + e_{15}\frac{\partial^2 w}{\partial x^2} + e_{33}\frac{\partial^2 w}{\partial z^2} - \varepsilon_{11}\frac{\partial^2 \varphi}{\partial x^2} - \varepsilon_{33}\frac{\partial^2 \varphi}{\partial z^2} &= 0. \end{aligned} \tag{2.3}$$

Due to the linearity of the Eqs. (2.1) – (2.3), the solution of the problem can be represented by a homogeneous electroelastic field and an additional field which ensures that the boundary conditions on the surface of the crack are fulfilled.

Applying the Fourier transform to Eqs. (2.3),

$$
\begin{aligned}
\overline{u}(p,z) &= \int_0^\infty u(x,z)\sin(px)dx,\\
\overline{w}(p,z) &= \int_0^\infty w(x,z)\cos(px)dx, \qquad (2.4)\\
\overline{\varphi}(p,z) &= \int_0^\infty \varphi(x,z)\cos(px)dx
\end{aligned}
$$

we arrive at a system of ordinary differential equations with respect $\overline{u}$, $\overline{w}$, $\overline{\varphi}$. The form of the solution of this system depends on the roots of the characteristic equation

$$
\det\|r_{ij}\| = 0, \qquad (i,j = 1,2,3), \tag{2.5}
$$

where

$$
\begin{aligned}
r_{11} &= c_{44}k^2 - c_{11}, \quad r_{12} = -r_{21} = k(c_{13} + c_{44}),\\
r_{13} &= -r_{31} = k(e_{31} + e_{15}),\\
r_{22} &= c_{33}k^2 - c_{44}, \quad r_{23} = r_{32} = e_{33}k^2 - e_{15}, \quad r_{33} = \varepsilon_{11} - \varepsilon_{33}k^2.
\end{aligned}
$$

Analysis of Eq. (2.5) shows that for realistic values of the material constants of the considered class of materials, this equation has two real roots $\pm k_1$ and two pairs of complex conjugate roots $\pm\delta \pm i\omega$ (see e.g. PARTON and KUDRYAVTSEV [24]). Therefore the solution of the system (2.3) can be represented in the form

$$
\begin{aligned}
u(x,z) &= \frac{2}{\pi}\int_0^\infty \Big\{\alpha_1 A(p)e^{-k_1pz} + [\alpha_2 B(p) - \alpha_3 C(p)]e^{-\delta pz}\cos(\omega pz)\\
&\qquad + [\alpha_3 B(p) + \alpha_2 C(p)]e^{-\delta pz}\sin(\omega pz)\Big\}\sin(px)dp,\\
w(x,z) &= \frac{2}{\pi}\int_0^\infty \Big\{\beta_1 A(p)e^{-k_1pz} + [\beta_2 B(p) - \beta_3 C(p)]e^{-\delta pz}\cos(\omega pz) \qquad (2.6)\\
&\qquad + [\beta_3 B(p) + \beta_2 C(p)]e^{-\delta pz}\sin(\omega pz)\Big\}\cos(px)dp,
\end{aligned}
$$

(2.6) [cont.]

$$\varphi(x,z) = \frac{2}{\pi}\int_0^\infty \Big\{\gamma_1 A(p)e^{-k_1 pz} + [\gamma_2 B(p) - \gamma_3 C(p)]e^{-\delta pz}\cos(\omega pz) + [\gamma_3 B(p) + \gamma_2 C(p)]e^{-\delta pz}\sin(\omega pz)\Big\}\cos(px)dp,$$

where

$$\alpha(k) = r_{12}r_{23} - r_{13}r_{22}, \quad \beta(k) = r_{13}r_{21} - r_{11}r_{23}, \quad \gamma(k) = r_{11}r_{22} - r_{12}r_{21},$$

$$\alpha_1 = \alpha(k_1), \quad \beta_1 = \beta(k_1), \quad \gamma_1 = \gamma(k_1),$$

$$\alpha_2 + i\alpha_3 = \alpha(\delta + i\omega), \quad \beta_2 + i\beta_3 = \beta(\delta + i\omega), \quad \gamma_2 + i\gamma_3 = \gamma(\delta + i\omega).$$

Taking into account these relations and the expressions (2.1), (2.2) for stresses and electric displacements via u, w, φ, one obtains at the interface $z = 0$ the following representations:

(2.7)

$$u^\pm(x,0) = \frac{2}{\pi}\int_0^\infty \left[\alpha_1^\pm A^\pm(p) + \alpha_2^\pm B^\pm(p) - \alpha_3^\pm C^\pm(p)\right]\sin(px)dp,$$

$$w^\pm(x,0) = \pm\frac{2}{\pi}\int_0^\infty \left[\beta_1^\pm A^\pm(p) + \beta_2^\pm B^\pm(p) - \beta_3^\pm C^\pm(p)\right]\cos(px)dp,$$

$$\varphi^\pm(x,0) = \pm\frac{2}{\pi}\int_0^\infty \left[\gamma_1^\pm A^\pm(p) + \gamma_2^\pm B^\pm(p) - \gamma_3^\pm C^\pm(p)\right]\cos(px)dp,$$

$$\sigma_{xz}^\pm(x,0) = \pm\frac{2}{\pi}\int_\pi^\infty \left[m_1^\pm A^\pm(p) + m_2^\pm B^\pm(p) - m_3^\pm C^\pm(p)\right]p\cdot\sin(px)dp,$$

$$\sigma_{zz}^\pm(x,0) = \frac{2}{\pi}\int_0^\infty \left[n_1^\pm A^\pm(p) + n_2^\pm B^\pm(p) - n_3^\pm C^\pm(p)\right]p\cdot\cos(px)dp + \sigma_{zz}^\infty,$$

$$D_z^\pm(x,0) = \frac{2}{\pi}\int_0^\infty \left[s_1^\pm A^\pm(p) + s_2^\pm B^\pm(p) - s_3^\pm C^\pm(p)\right]p\cdot\cos(px)dp + D_z^\infty.$$

The functions $A^{\pm}(p)$, $B^{\pm}(p)$, $C^{\pm}(p)$ are to be found from the boundary conditions. Expressions for $m_i^{\pm}$, $n_i^{\pm}$, $s_i^{\pm}$ $(i = 1, 2, 3)$ are given in the Appendix.

The conditions to be satisfied at the interface are:

$$\sigma_{xz}^{+}(x,0) = \sigma_{xz}^{-}(x,0), \quad \sigma_{zz}^{+}(x,0) = \sigma_{zz}^{-}(x,0), \quad D_z^{+}(x,0) = D_z^{-}(x,0). \tag{2.8}$$

Substituting the formulas (2.7) into (2.8), we obtain the relations

$$\begin{aligned} A^{-}(t) &= d_{11}A^{+}(t) + d_{12}B^{+}(t) + d_{13}C^{+}(t),\\ B^{-}(t) &= d_{21}A^{+}(t) + d_{22}B^{+}(t) + d_{23}C^{+}(t),\\ C^{-}(t) &= d_{31}A^{+}(t) + d_{32}B^{+}(t) + d_{33}C^{+}(t), \end{aligned}$$

where the constants d_{ij} $(i, j = 1, 2, 3)$ are given in the Appendix.

Next, we introduce the unknown functions

$$\begin{aligned} g_1(x) &= \sigma_{xz}^{+}(x,0) = \sigma_{xz}^{-}(x,0),\\ g_2(x) &= \frac{\partial w^{+}(x,0)}{\partial x} - \frac{\partial w^{-}(x,0)}{\partial x},\\ g_3(x) &= \frac{\partial \varphi^{+}(x,0)}{\partial x} - \frac{\partial \varphi^{-}(x,0)}{\partial x}, \end{aligned} \tag{2.9}$$

and apply the Fourier transform to the expressions obtained after substituting $\sigma_{xz}^{\pm}(x,0)$, $w^{\pm}(x,0)$, $\varphi^{\pm}(x,0)$ from (2.7) into (2.9). This leads to a system of linear algebraic equations with respect to $A^{+}(t)$, $B^{+}(t)$ and $C^{+}(t)$ which has the following solutions:

$$\begin{aligned} A^{+}(t) &= \frac{1}{t}\left[a_{11}\overline{g}_1(t) + a_{12}\overline{g}_2(t) + a_{13}\overline{g}_3(t)\right],\\ B^{+}(t) &= \frac{1}{t}\left[a_{21}\overline{g}_1(t) + a_{22}\overline{g}_2(t) + a_{23}\overline{g}_3(t)\right],\\ C^{+}(t) &= \frac{1}{t}\left[a_{31}\overline{g}_1(t) + a_{32}\overline{g}_2(t) + a_{33}\overline{g}_3(t)\right], \end{aligned} \tag{2.10}$$

where

$$\overline{g}_k(t) = \int_0^{\infty} g_k(x)\sin(tx)\,dx, \quad k = 1, 2, 3,$$

and the constants $a_{ij} (i, j = 1, 2, 3)$ are given in the Appendix.

Substituting the formulas (2.10) into expressions for $\Delta(x,0) = \dfrac{\partial u^+(x,0)}{\partial x} - \dfrac{\partial u^-(x,0)}{\partial x}$, $\sigma_{zz}^+(x,0)$, $D_z^+(x,0)$ from (2.7) and taking into account that

$$\frac{2}{\pi}\int_0^\infty \bar{g}_k(t)\cos(tx)dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t-x}g_k(t)dt, \qquad k=1,2,3,$$

we arrive at the boundary relations

$$\Delta(x,0) = \frac{1}{\pi}\left\{\eta_{11}\int_{-\infty}^{\infty}\frac{g_1(t)dt}{t-x} + \eta_{12}\int_{-\infty}^{\infty}\frac{g_2(t)dt}{t-x} + \eta_{13}\int_{-\infty}^{\infty}\frac{g_3(t)dt}{t-x}\right\} + \Omega_1, \tag{2.11}$$

$$\sigma_{zz}^+(x,0) = \frac{1}{\pi}\left\{\eta_{21}\int_{-\infty}^{\infty}\frac{g_1(t)dt}{t-x} + \eta_{22}\int_{-\infty}^{\infty}\frac{g_2(t)dt}{t-x} + \eta_{23}\int_{-\infty}^{\infty}\frac{g_3(t)dt}{t-x}\right\} + \Omega_2, \tag{2.12}$$

$$D_z^+(x,0) = \frac{1}{\pi}\left\{\eta_{31}\int_{-\infty}^{\infty}\frac{g_1(t)dt}{t-x} + \eta_{32}\int_{-\infty}^{\infty}\frac{g_2(t)dt}{t-x} + \eta_{33}\int_{-\infty}^{\infty}\frac{g_3(t)dt}{t-x}\right\} + \Omega_3, \tag{2.13}$$

where $\Omega_1 = 0$, $\Omega_2 = \sigma_{zz}^\infty$, $\Omega_3 = D_z^\infty$, and the formulas for the constants $\eta_{ij}(i,j = 1,2,3)$ are given in the Appendix.

The boundary integral relations (2.11) – (2.13) play an important role in the following analysis because by means of these relations the singular integral equations for various mixed boundary conditions at the interface can be formulated.

3. Interface crack with permeable crack faces

We now turn to the permeable crack faces. Fracture mechanical parameters calculated by taking into account the permittivity of the crack medium, indicate that permeable crack faces are more realistic extreme case (DUNN [14], BALKE *et al.* [16], GRUEBNER and KAMLAH [17], MCMEEKING [20]).

So we now assume that the complete interface $z = 0$ is electrically permeable. In this case, the boundary conditions are the following

$$
\begin{aligned}
&\sigma_{xz}^{+}(x,0) = 0, \quad \sigma_{zz}^{+}(x,0) = 0, \quad |x| < b;\\
&u^{+}(x,0) - u^{-}(x,0) = 0, \quad w^{+}(x,0) - w^{-}(x,0) = 0, \quad |x| > b;\\
&\varphi^{+}(x,0) - \varphi^{-}(x,0) = 0, \quad |x| < \infty.
\end{aligned}
\tag{3.1}
$$

Taking into account that according to (3.1), $g_1(x) = 0$ for $|x| < b$, $g_2(x) = 0$ for $|x| > b$, $g_3(x) \equiv 0$, and by satisfying the equation $\Delta(x,0) = 0$ as well as the second of the conditions (3.1), we arrive at the following system of singular integral equations (SIE):

$$
\eta_{i1} \int\limits_{c_1}^{d_1} \left[\frac{1}{t-x} + \frac{1}{t+x} \right] g_1(t)dt + \eta_{i2} \int\limits_{c_2}^{d_2} \frac{1}{t-x} g_2(t)dt = -\pi \Omega_i. \tag{3.2}
$$

Here, $x \in (c_i, d_i)$, $c_1 = -c_2 = d_2 = b$, $d_1 = \infty$, $i = 1, 2$.

In this case we obtain singularities which imply oscillations in the stress field near the tips of the crack, resulting in the unrealistic phenomenon of interpenetrating of crack faces, and leading to difficulties in the numerical analysis for the determination of the fracture mechanical parameters. For this reason, similar to COMNINOU [8] we introduce artificial frictionless contact zones $a < |x| < b$ to avoid the oscillating singularity, where the position of the point a is arbitrary for the time being. For an arbitrary position of the point a, an artificial contact zone model (ACZM) is not physically justified, but from this model the specific value of a for the real contact zone length (Comninou's model) can be found. Moreover, the quasi-invariance of the energy release rate (ERR) with respect to a, demonstrated later on for an ACZM can be used for a numerical (FEM or BEM) investigation of finite sized piezoelectrics with interface cracks.

The boundary conditions for considered model are the following:

$$
\begin{aligned}
&\sigma_{xz}^{+}(x,0) = 0, \quad \sigma_{zz}^{+}(x,0) = 0, \quad |x| < a;\\
&\sigma_{xz}^{+}(x,0) = 0, \quad w^{+}(x,0) - w^{-}(x,0) = 0, \quad a < |x| < b, |x| > h;\\
&u^{+}(x,0) - u^{-}(x,0) = 0, \quad w^{+}(x,0) - w^{-}(x,0) = 0, \quad b < |x| < h;\\
&\varphi^{+}(x,0) - \varphi^{-}(x,0) = 0, \quad |x| < \infty.
\end{aligned}
\tag{3.3}
$$

By these conditions, further zones of frictionless contact have been introduced for $|x| > h$. This assumption is not a principal one, since according to St.Venant's principle, it will not influence the state of stress near the crack, provided $h \gg b$.

However, in this case $g_1(x) = 0$ for $|x| < b$ and $|x| > h$, $g_2(x) = 0$ for $|x| > a$, $g_3(x) \equiv 0$ and from equation $\Delta(x,0) = 0$ and the second of the conditions (16), we arrive at the very same system of SIE (3.2), in which now $x \in (c_i, d_i)$, $c_1 = b$, $d_1 = h$, $c_2 = -a$, $d_2 = a$, $i = 1, 2$. Note that we have finite upper limits in the integrals due to the further frictionless contact zones.

Additional conditions which must be satisfied to ensure that the displacements are single valued are (see e.g. COMNINOU and DUNDURS [25])

$$\int_{-b}^{b} \Delta(x,0)dx = 0, \qquad \int_{-a}^{a} \left(\frac{\partial w^+(x,0)}{\partial x} - \frac{\partial w^-(x,0)}{\partial x} \right) dx = 0,$$

which, after substitution of relation (2.7) can be written in the form

$$2\eta_{11} \int_{b}^{h} \ln \left| \frac{t+b}{t-b} \right| g_1(t)dt + \eta_{12} \int_{-a}^{a} \ln \left| \frac{t+b}{t-b} \right| g_2(t)dt = 0, \tag{3.4}$$

$$\int_{-a}^{a} g_2(t)dt = 0.$$

Due to the absence of an oscillation in the singularity we can represent in this case the unknown functions $g_i(t)(i = 1, 2)$ in the form

$$g_i(t) = \frac{g_i^*(t)}{\sqrt{(d_i - t)(t - c_i)}}, \qquad g_i^*(t) \in H, \tag{3.5}$$

where H is the class of Hölder functions (see e.g. MUSKHELISHVILI [26]) in $[c_i, d_i]$, $i = 1, 2$.

Next, we introduce the stress intensity factors for the right-hand crack tip by

$$K_1 = \lim_{x \to a+0} \sqrt{2(x-a)}\sigma_{zz}^+(x,0), \quad K_2 = \lim_{x \to b+0} \sqrt{2(x-b)}\sigma_{xz}^+(x,0). \tag{3.6}$$

Due to (2.12), (3.5) and the results of MUSKHELISHVILI [26] for Cauchy type integrals behaviour, we can write

$$K_1 = -\frac{\eta_{22} g_2^*(a)}{\sqrt{a}}, \qquad K_2 = \sqrt{\frac{2}{h-b}} g_1^*(b). \tag{3.7}$$

The energy release rate G for the right crack tip can be represented by the virtual work integral

$$G = \lim_{\Delta l \to 0} \left\{ \frac{1}{2\Delta l} \int\limits_{a}^{a+\Delta l} \tilde{\sigma}_{zz}^{+}(x)\tilde{w}(x+\Delta l)dx + \frac{1}{2\Delta l} \int\limits_{b}^{b+\Delta l} \tilde{\sigma}_{xz}^{+}(x)\tilde{u}(x+\Delta l)dx \right\}. \tag{3.8}$$

The integrals in the last formula can be computed from the asymptotic behaviour of displacements and stresses near the singular points a and b.

Exploiting the asymptotic behaviour

$$g_2(x)\big|_{x\to a-0} = \frac{g_2^*(a)}{\sqrt{2a(a-x)}}$$

following from (3.5), the second of Eq. (2.9) gives

$$\left\{w^+(x,0) - w^-(x,0)\right\}\big|_{x\to a-0} = -\sqrt{\frac{2(a-x)}{a}}\, g_2^*(a).$$

Employing Cauchy-type integral properties [26] in Eq. (2.12) gives

$$\sigma_{zz}^{+}(x,0)\big|_{x\to a+0} = -\eta_{22}\frac{g_2^*(a)}{\sqrt{2a(x-a)}}.$$

Using the first of the relation (3.7) leads to the following formulas:

$$\tilde{w}(x) = \left\{w^+(x,0) - w^-(x,0)\right\}\big|_{x\to a-0} = \frac{\sqrt{2(a-x)}}{\eta_{22}}K_1,$$

$$\tilde{\sigma}_{zz}^{+}(x) = \sigma_{zz}^{+}(x,0)\big|_{x\to a+0} = \frac{K_1}{\sqrt{2(x-a)}}.$$

In a similar way it can be shown that

$$\tilde{u}(x) = \left\{u^+(x,0) - u^-(x,0)\right\}\big|_{x\to b-0} = -\eta_{11}\sqrt{2(b-x)}K_2,$$

$$\tilde{\sigma}_{xz}^{+}(x) = \sigma_{xz}^{+}(x,0)\big|_{x\to b+0} = \frac{K_2}{\sqrt{2(x-b)}}.$$

Substituting the last formulas into (3.8) and taking into account that

$$\int\limits_{0}^{\Delta l} \sqrt{\frac{\Delta l - x}{x}}\,dx = \frac{\pi \Delta l}{2},$$

we obtain the following formula:

$$G = \frac{\pi}{4}\left[\overline{\alpha}K_1^2 + \overline{\beta}K_2^2\right], \quad \overline{\alpha} = 1/\eta_{22}, \quad \overline{\beta} = -\eta_{11}, \quad (\eta_{22} > 0,\ \eta_{11} < 0). \tag{3.9}$$

4. Interface crack with insulated crack faces

We now consider the second extreme case of electric boundary conditions, namely electrically insulating crack faces. Sometimes, they are thought to be more realistic since the permittivity of piezoceramics is three orders of magnitude higher than the one of air or vacuum (SUO *et al.* [18], PARK and SUN [19]). Thus we assume next that the crack faces are perfectly insulated. The boundary conditions for this case are the following:

$$\begin{aligned}
&\sigma^+_{xz}(x,0) = 0, \quad \sigma^+_{zz}(x,0) = 0, \quad |x| < b;\\
&u^+(x,0) - u^-(x,0) = 0, \quad w^+(x,0) - w^-(x,0) = 0, \quad b < |x| < h;\\
&\sigma^+_{xz}(x,0) = 0, \quad w^+(x,0) - w^-(x,0) = 0, \quad |x| > h;\\
&D^+_z(x,0) = 0, \quad |x| < b, \quad \varphi^+(x,0) - \varphi^-(x,0) = 0, \quad |x| > b.
\end{aligned} \tag{4.1}$$

Taking into account that in this case $g_1(x) = 0$, $(|x| < b, |x| > h)$, $g_2(x) = 0$ $(|x| > b)$, and $g_3(x) = 0$ $(|x| > b)$, we satisfy by means of relations (2.11) – (2.13) the relation $\Delta(x,0) = 0$ $(b < |x| < h)$, the second, and the last but one of the boundary conditions (4.1), and we arrive at the following system of three SIE:

$$\eta_{i1} \int\limits_b^h \left[\frac{1}{t-x} + \frac{1}{t+x}\right] g_1(t)dt + \sum_{k=2}^{3} \eta_{ik} \int\limits_{-b}^{b} \frac{1}{t-x} g_k(t)dt = -\pi\Omega_i, \tag{4.2}$$

where $x \in (b,h)$ for $i = 1$ and $x \in (-b,b)$ for $i = 2,3$.

Additional conditions which ensure that the displacements and the electrical potential are single valued, are taken to be in the form

$$2\eta_{11} \int\limits_b^h \ln\left|\frac{t+b}{t-b}\right| g_1(t)dt + \sum_{k=2}^{3} \eta_{1k} \int\limits_{-b}^{b} \ln\left|\frac{t+b}{t-b}\right| g_k(t)dt = 0, \tag{4.3}$$

$$\int\limits_{-b}^{b} g_k(t)dt = 0, \qquad (k = 2,3).$$

We represent the unknown functions as

$$g_1(t) = \frac{g_1^*(t)}{(t-b)^\alpha \sqrt{h-t}}, \quad g_k(t) = \frac{g_k^*(t)}{(b-t)^\alpha (b+t)^\alpha}, \quad (k = 2,3), \tag{4.4}$$

where $g_i^*(t) \in H$ $(i = 1,2,3)$ and $0 \le \mathrm{Re}(\alpha) < 1$.

Introducing the piecewise holomorphic functions

$$\Phi_1(x) = \frac{1}{\pi}\int\limits_b^h \frac{g_1(t)}{t-x}dt, \qquad \Phi_k(x) = \frac{1}{\pi}\int\limits_{-b}^{b} \frac{g_k(t)}{t-x}dt, \quad (k=2,3),$$

one can rewrite Eq. (4.2) as

$$\eta_{i1}\left[\Phi_1(x)+\Phi_1(-x)\right] + \sum_{k=2}^{3}\eta_{ik}\Phi_k(x) = -\Omega_i. \tag{4.5}$$

Using the approach described in the paper of MUSKHELISHVILI [26] for estimating Cauchy type integrals at the boundary points of the integration interval, one obtains for $x \to b+0$ and $x \to b-0$

$$\begin{aligned}
\Phi_1(x)|_{x\to b+0} &= \frac{\operatorname{ctg}(\pi\alpha)g_1^*(b)}{(x-b)^\alpha\sqrt{h-b}} + \Phi_1^*(x),\\
\Phi_1(x)|_{x\to b-0} &= \frac{g_1^*(b)}{\sin(\pi\alpha)(b-x)^\alpha\sqrt{h-b}} + \Phi_1^*(x),\\
\Phi_k|_{x\to b+0} &= -\frac{g_k^*(b)}{\sin(\pi\alpha)(x-b)^\alpha(2b)^\alpha} + \Phi_k^*(x),\\
\Phi_k(x)|_{x\to b-0} &= -\frac{\operatorname{ctg}(\pi\alpha)g_k^*(b)}{(2b)^\alpha(b-x)^\alpha} + \Phi_k^*(x).
\end{aligned} \tag{4.6}$$

Here $|\Phi_i^*(x)| \le \dfrac{C_i}{(x-b)^\gamma}$ with $\operatorname{Re}(\gamma) < \operatorname{Re}(\alpha)$ and $C_i (i=1,2,3)$ are real constants.

Substituting (4.6) into (4.5), we arrive to the following equations:

$$\begin{aligned}
\frac{\eta_{11}\cos(\pi\alpha)g_1^*(b)}{\sqrt{h-b}} - \frac{\eta_{12}g_2^*(b)}{(2b)^\alpha} - \frac{\eta_{13}g_3^*(b)}{(2b)^\alpha} &= (x-b)^\alpha\Phi_4^*(x),\\
\frac{\eta_{21}g_1^*(b)}{\sqrt{h-b}} - \frac{\eta_{22}\cos(\pi\alpha)g_2^*(b)}{(2b)^\alpha} - \frac{\eta_{23}\cos(\pi\alpha)g_3^*(b)}{(2b)^\alpha} &= (b-x)^\alpha\Phi_5^*(x),\\
\frac{\eta_{31}g_1^*(b)}{\sqrt{h-b}} - \frac{\eta_{32}\cos(\pi\alpha)g_2^*(b)}{(2b)^\alpha} - \frac{\eta_{33}\cos(\pi\alpha)g_3^*(b)}{(2b)^\alpha} &= (b-x)^\alpha\Phi_6^*(x).
\end{aligned}$$

This system has nonzero solutions for if

$$\det\|b_{ij}\| = 0, \qquad (i,j=1,2,3),$$

where

$$b_{11} = \eta_{11}\cos(\pi\alpha)/\sqrt{h-b}, \quad b_{12} = -\eta_{12}/(2b)^{\alpha}, \quad b_{13} = -\eta_{13}/(2b)^{\alpha},$$

$$b_{21} = \eta_{21}/\sqrt{h-b}, \quad b_{22} = -\eta_{22}\cos(\pi\alpha)/(2b)^{\alpha},$$

$$b_{23} = -\eta_{23}\cos(\pi\alpha)/(2b)^{\alpha}, \quad b_{31} = \eta_{31}/\sqrt{h-b},$$

$$b_{32} = -\eta_{32}\cos(\pi\alpha)/(2b)^{\alpha}, \quad b_{33} = -\eta_{33}\cos(\pi\alpha)/(2b)^{\alpha}.$$

This determinant can be rewritten in the form

$$\cos(\pi\alpha)\left[\mu_1\cos^2(\pi\alpha) + \mu_2\right] = 0, \tag{4.7}$$

where

$$\mu_1 = \eta_{11}\eta_{22}\eta_{33} - \eta_{11}\eta_{32}\eta_{23},$$

$$\mu_2 = \eta_{21}\eta_{32}\eta_{13} + \eta_{12}\eta_{23}\eta_{31} - \eta_{31}\eta_{22}\eta_{13} - \eta_{21}\eta_{12}\eta_{33}.$$

Equation (4.7) can be used for the determination of the power of singularity at the tip of interface crack for insulated crack surfaces.

5. Numerical results and discussion

First of all we pay our attention to the ACZM for the permeable crack faces. The values of the material constants of the piezoceramics PZT-4, PZT-19, PZT-5H, PZT-5 were adopted from the papers [19, 24, 27, 33], respectively.

A numerical solution of the system (3.2), (3.4) has been obtained by the method based on the Gauss-Chebyshev quadrature rule [28] which was described in detail in the paper of Loboda [29] for equations similar to (3.2), (3.4).

The variation of the normal stress in the contact region corresponding to different $\lambda = \dfrac{b-a}{2b}$ are shown in Fig. 2. For the upper piezomaterial, fictitious material constants were taken as

$$c_{11} = 1.22\cdot 10^{10}(\mathrm{N/m^2}), \quad c_{12} = 7.78\cdot 10^{10}(\mathrm{N/m^2}),$$

$$c_{13} = 0.11\cdot 10^{10}(\mathrm{N/m^2}), \quad c_{33} = 1.23\cdot 10^{10}(\mathrm{N/m^2}),$$

$$c_{44} = 1.0\cdot 10^{10}(\mathrm{N/m^2}), \quad e_{31} = -6.50(\mathrm{C/m^2}),$$

$$e_{33} = 23.3(\mathrm{C/m^2}), \quad e_{15} = 17.0(\mathrm{C/m^2}),$$

$$\varepsilon_{11} = 15.1\cdot 10^{-9}(\mathrm{C/Vm}), \quad \varepsilon_{33} = 13.0\cdot 10^{-9}(\mathrm{C/Vm}),$$

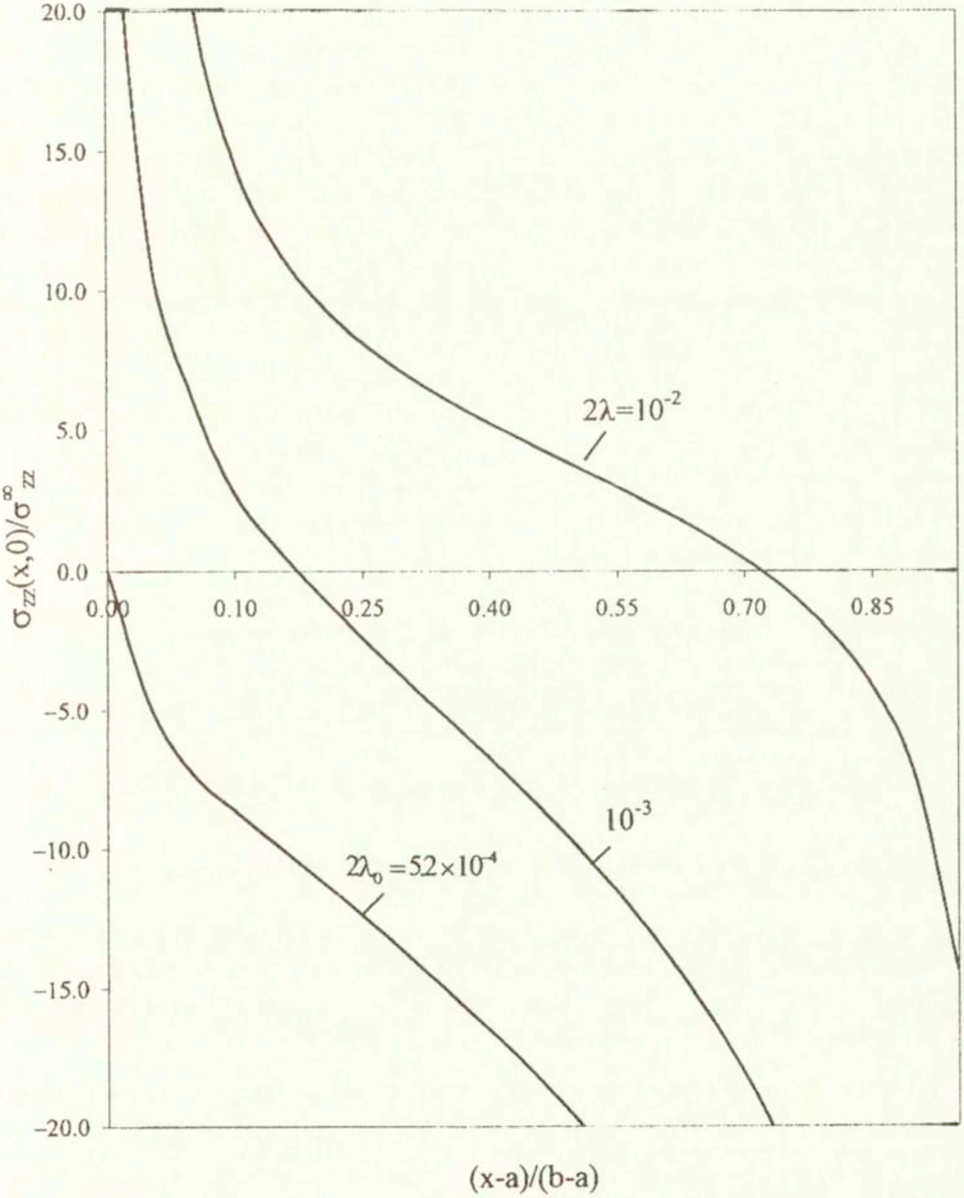

FIG. 2. Variation of the normal stress along the contact zone $[a,b]$ for different values of λ.

and PZT-4 was chosen as the lower piezomaterial. One can see that for sufficiently large λ, the values of $\sigma_{zz}(x,0)$ in the main part of $[a,b]$ are positive. Only at point b they are negative indicating a compressive stress. Decreasing λ leads to an increased relative length of the compressed zone with respect to the length of $[a,b]$. Finally, for $\lambda = \lambda_0 = 2.6\times10^{-4}$ the values of $\sigma_{zz}(x,0)$ becomes compressive in the complete contact region $[a,b]$. This means that the crack faces are in contact in all of $[a,b]$ and the boundary condition $w^+(x,0) - w^-(x,0) = 0$ for $x \in [a,b]$ is physically correct for $\lambda = \lambda_0$. In this sense, λ_0 is the real contact zone length.

Next, the values of K_1, K_2 and G for the same piezomaterials as before and various λ are shown in Table 1. The last line corresponds to the real contact zone length λ_0 and the maximum value of $|K_2|$ and $K_1 = 0$, i.e $g_2^*(a) = 0$, are found for $\lambda = \lambda_0$. This value for λ_0 is rather small (even for the considered fictitious upper piezomaterial) and it can not be found easily for a finite-sized body. But fortunately this is not necessary because the ERR $G(\lambda)$ is almost invariable for $\lambda_0 \le \lambda \le \lambda_*$ ($\lambda_* \sim 5\times10^{-2}$), and similarly to LOBODA [30] one may consider it as quasi-invariant with respect to λ. This fact permits to solve a problem in question for $\lambda = \lambda_* \gg \lambda_0$ and use the obtained value of G as $G_0 = G(\lambda_0)$. This approach simplifies the numerical solution of an interface crack problem for piezoelectric composites significantly.

Table 1. Dependence of the SIF-s K_i and the ERR G on the relative contact zone length λ for electrically permeable crack.

2λ	$K_2(\mathrm{Pa}\sqrt{m})$	$K_1(\mathrm{Pa}\sqrt{m})$	$G(\mathrm{N/m})$
5×10^{-2}	-0.861	1.016	0.165×10^{-9}
10^{-2}	-1.214	0.698	0.167×10^{-9}
5×10^{-3}	-1.319	0.531	0.167×10^{-9}
10^{-3}	-1.441	0.134	0.167×10^{-9}
$2\lambda_0\approx5.2\times10^{-4}$	-1.445	≈ 0	0.166×10^{-9}

The quasi-invariance of G with respect to λ is confirmed in Table 2 which has been obtained for the ceramics PZT-5H/PZT-4. Due to the decreasing K_1 and increasing K_2, it is reasonable to assume for this combination of materials the existence of $\lambda = \lambda_0$ for which $K_1 = 0$, too.

Table 2. Dependence of the SIFs K_i and the ERR G on the relative contact zone length λ for electrically permeable crack for the piezoceramics PZT-5H/PZT-4.

2λ	$K_2(\mathrm{Pa}\sqrt{m})$	$K_1(\mathrm{Pa}\sqrt{m})$	$G(\mathrm{N/m})$
10^{-2}	-0.106	0.986	0.331×10^{-10}
10^{-3}	-0.153	0.985	0.332×10^{-10}
10^{-4}	-0.178	0.982	0.332×10^{-10}
10^{-5}	-0.182	0.981	0.332×10^{-10}

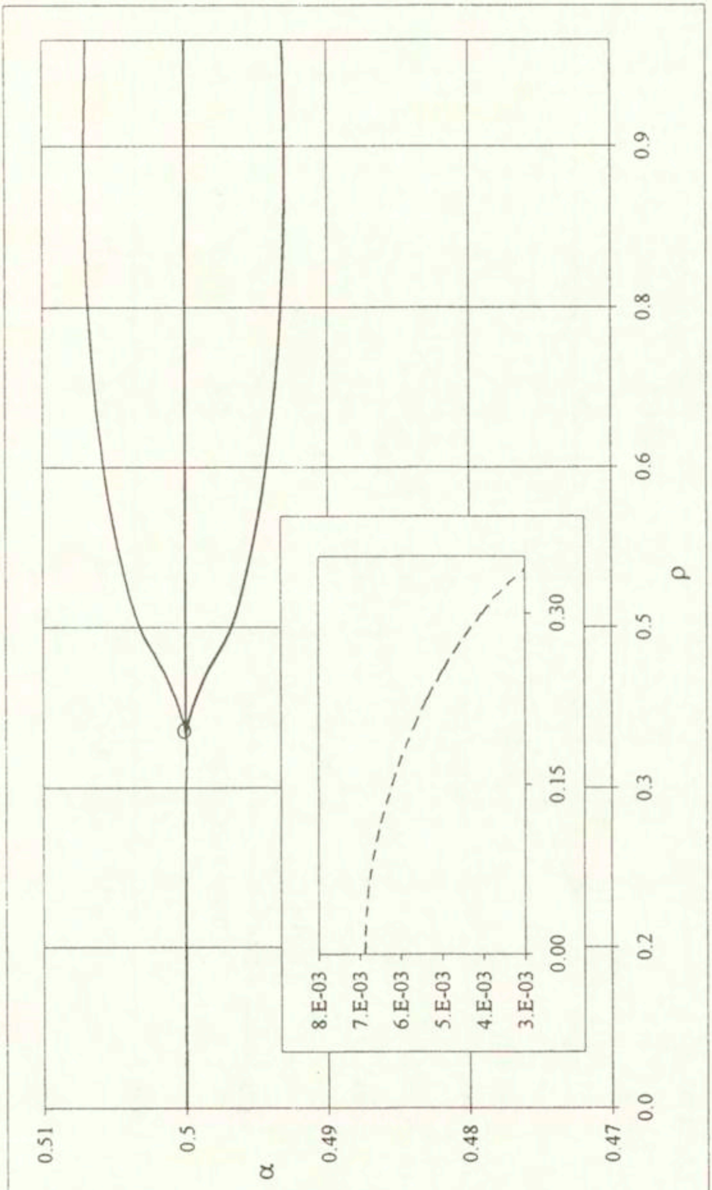

FIG. 3. Dependence of the power of singularity on the piezoelectric constants. The dielectric and elastic constants were chosen according to the combination of PZT-19/ PZT-5 piezoceramics. The solid lines represent the real and the dashed lines the imaginary part, respectively.

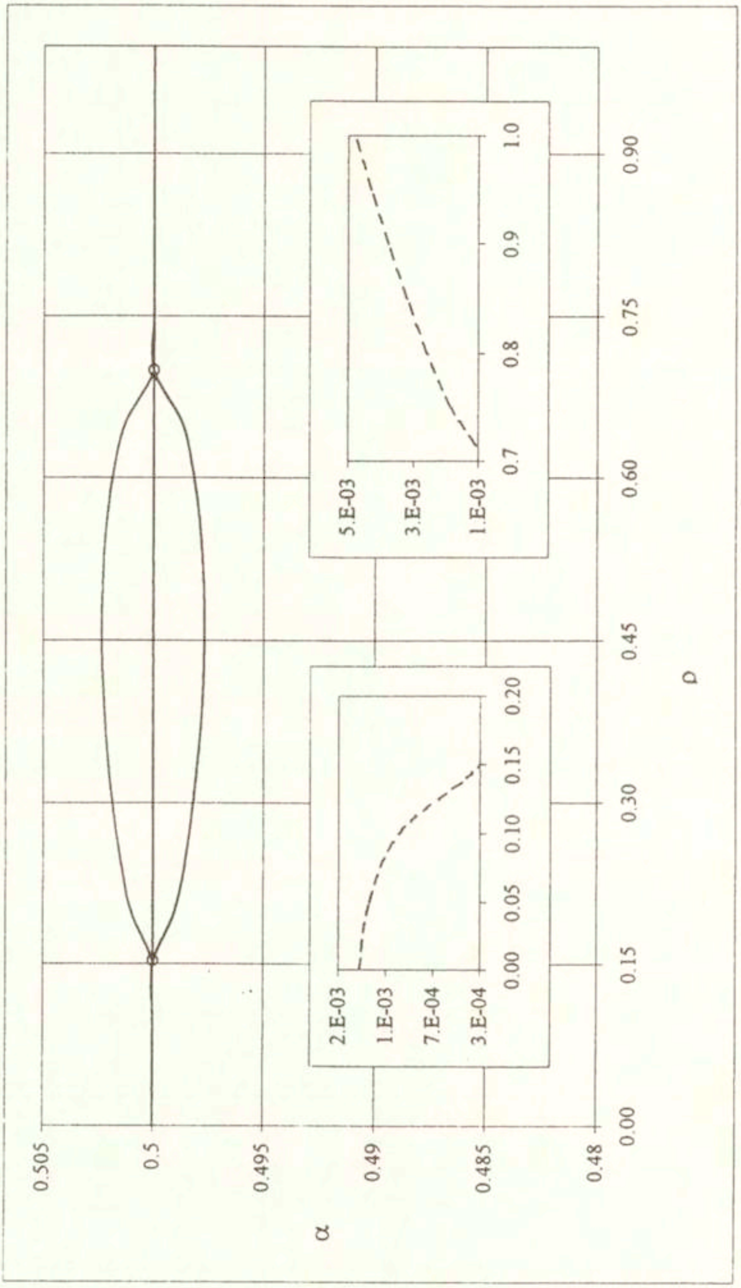

FIG. 4. Dependence of the power of singularity on the piezoelectric constants. The dielectric and elastic constants were chosen according to the combination of PZT-4/PZT-5 piezoceramics. The solid lines represent the real and the dashed lines the imaginary part, respectively.

Next, we consider the numerical solution of the system (4.2), (4.3) for insulated crack faces.

For this purpose the roots of the Eq. (4.7) were found numerically. Some of the most interesting results are shown in Fig. 3 and Fig. 4 where the dependence of the roots of Eq. (4.7) on the parameter ρ is displayed. Parameter ρ is introduced in order to investigate the influence of a varying degree of piezoelectric coupling in both piezomaterials: $\hat{e}_{31}^{\pm} = \rho e_{31}^{\pm}$, $\hat{e}_{33}^{\pm} = \rho e_{33}^{\pm}$, $\hat{e}_{15}^{\pm} = \rho e_{15}^{\pm}$. As ρ is increased from 0 to 1, the piezocoefficients in both the upper and the lower piezomaterials are increased from zero to their real value: $\hat{e}_{ij}^{\pm} = 0.0...e_{ij}^{\pm}$ for $\rho = 0.0...1.0$. The coefficients of $c_{11}^{\pm}$, $c_{13}^{\pm}$, $c_{33}^{\pm}$, $c_{44}^{\pm}$, $e_{31}^{\pm}$, $e_{33}^{\pm}$, $e_{15}^{\pm}$, $\varepsilon_{11}^{\pm}$, $\varepsilon_{33}^{\pm}$ were chosen according to piezomaterials PZT-19/PZT-5 (Fig. 3) and PZT-4/PZT-5 (Fig. 4), respectively. The rigid lines are related to the real parts of the roots while the dashed lines represent the corresponding imaginary parts (0.5 is a real root for any ρ). Circles on these lines indicate the transition from complex to real roots.

These results confirm the conclusion obtained earlier in another way in the papers [18, 22] concerning the possibility of a non-oscillating singularity for an interface crack in a piezomaterial compound. Here, we demonstrate additionally the rather complicated dependence of the power of singularity on the piezoelectric parameters and the appearance of both the real singularity for the ceramics PZT-19/PZT-5 (Fig. 3, $\rho = 1$) and the oscillating singularity for the ceramics PZT-4/PZT-5 (Fig. 4, $\rho = 1$).

The numerical solution of the system (4.2), (4.3) for the case of a real non-square root singularity was found by the numerical method based upon the Gauss-Jacobi quadrature rule suggested by ERDOGAN *et al.* [31] and described with some modifications by GOVORUKHA and LOBODA [32].

In Fig. 5 and Fig. 6, the variations of $\sigma_{zz}(x,0)$ and $D_z(x,0)$ respectively in the neighbourhood next to the right-hand crack tip are shown. The material parameters were taken for the combination PZT-19/PZT-4 and the crack faces were chosen to be perfectly insulated. The lines I are related to the remote tensile load $\sigma_{zz}^{\infty} = 1.0$ (MPa) while the lines II are related to the remote electric flux with $D_z^{\infty} = 1.0$ (C/m^2) (Fig. 5) and $D_z^{\infty} = 1.0(\times 10^{-8}$ C/m^2) (Fig. 6). The complicated behaviour of the fields at the crack tip can be seen, which can be explained by the influence of singularity occurring in this point. But it should be noted that for increasing x, the values of $\sigma_{zz}(x,0)$ and $D_z(x,0)$ rather stable tend to their values at infinity. This fact confirms the high accuracy of the applied analytical approach and the numerical method.

Finally, in Fig. 7 the values of $D_z(x,0)$ for the special case of a homogeneous material PZT-4 are shown. The solid line corresponds to permeable crack faces, the dashed line to – insulated crack faces. For comparison, the values of $D_z(x,0)$ calculated according to the analytical solution (see e.g. PARTON and

KUDRYAVTSEV [24]) are shown as well (such analytical solutions are only found for homogeneous materials). It is obvious that these values completely coincide with the numerical results obtained by the method developed in this paper. Note that in contrast to the mentioned analytical solution, the validity of our solution is not restricted to the neighbourhood of the crack tip.

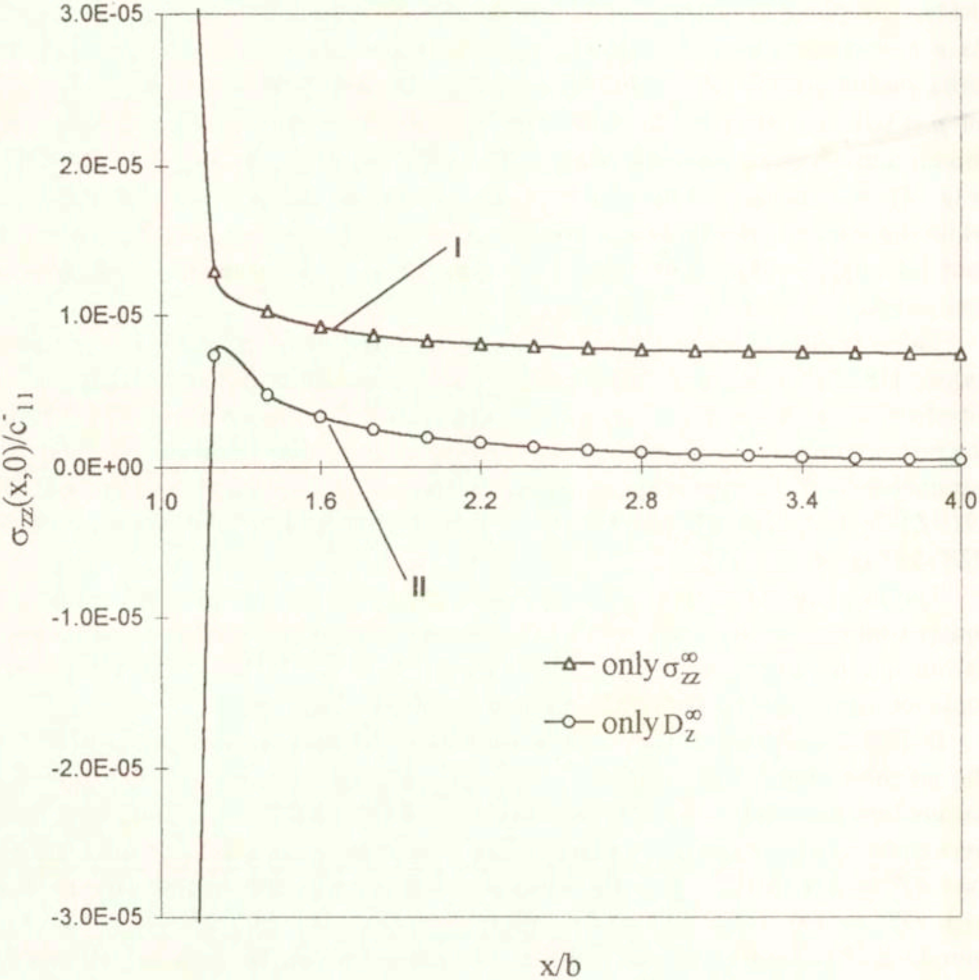

FIG. 5. Variations of the normal stresses for mechanical (line I) and electric (line II) loading.

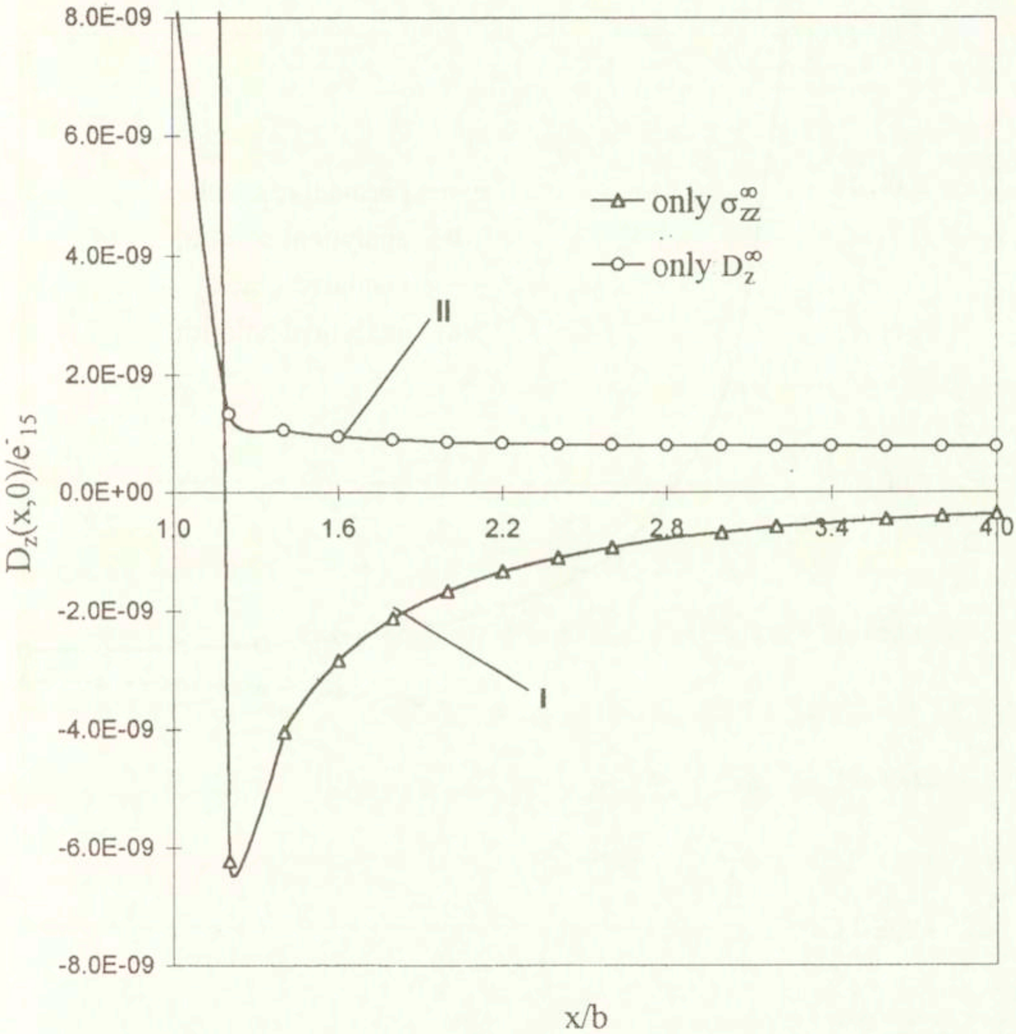

FIG. 6. Variations of the electric displacement for mechanical (line I) and electric (line II) loading.

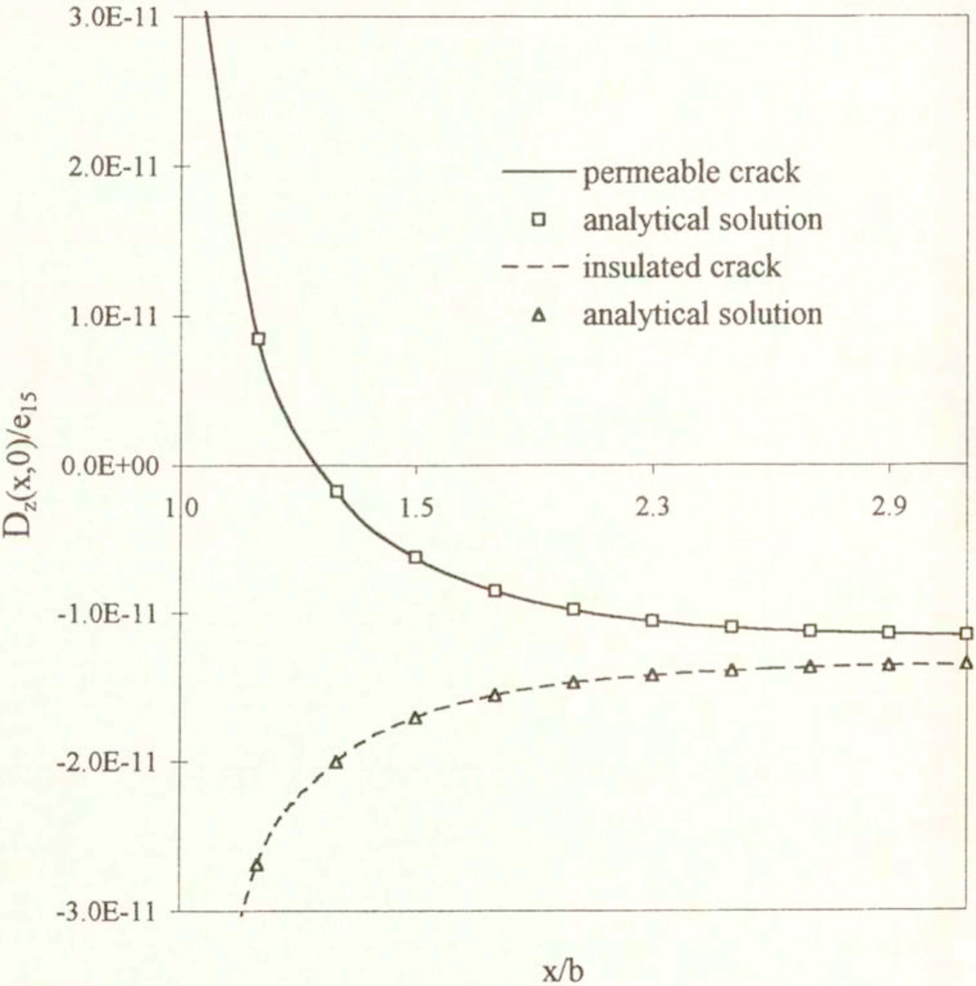

FIG. 7. Electric displacement for a crack in a homogeneous material. The solid line (our results) and marked line (analytical solution) – for electrically permeable crack; dashed line (our results) and marked line (analytical solution) – for insulated crack faces.

6. Conclusion

An interface crack between two piezoelectric semi-infinite planes is considered. The poling axis is assumed to be orthogonal to the crack plane and remote homogeneous tensile stresses and electric displacements act in infinite distance.

The methods of Fourier transforms and singular integral equations are utilised and both permeable and insulated crack faces are considered. For the first case, in addition to the open crack model, the contact zone model with an artificial contact zone is considered. Comninou's model with real contact zone is obtained as a particular case of this model and the quasi-invariance of the energy release rate with respect to the contact zone length is shown. The last property is rather useful from the point of view of the application of numerical methods for the investigation of finite-sized piezoelectric compounds with interface cracks because it permits to find the energy release rate for an artificial contact zone, and use approximately this value as the needed energy release rate.

The dependence of the power of singularity at a crack tip on the materials properties for perfectly insulated crack faces is investigated. The complete disappearance of an oscillating singularity in exchange for a real power singularity for most real ceramics of the considered class was revealed. The behaviour of stresses at the crack tip for both types of crack face conditions is obtained numerically and compared to an exact analytical solution for the special case of homogeneous materials.

Acknowledgment

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Appendix

$$
\begin{aligned}
m_1^{\pm} &= -e_{15}^{\pm}\gamma_1^{\pm} - c_{44}^{\pm}(\alpha_1^{\pm}k_1^{\pm} + \beta_1^{\pm}),\\
m_2^{\pm} &= -e_{15}^{\pm}\gamma_{21}^{\pm} - c_{44}^{\pm}(\alpha_2^{\pm}\delta^{\pm} - \alpha_3^{\pm}\omega^{\pm} + \beta_2^{\pm}),\\
m_3^{\pm} &= -e_{15}^{\pm}\gamma_3^{\pm} - c_{44}^{\pm}(\alpha_2^{\pm}\omega^{\pm} + \alpha_3^{\pm}\delta^{\pm} + \beta_3^{\pm}),\\
n_1^{\pm} &= c_{13}^{\pm}\alpha_1^{\pm} - c_{33}^{\pm}\beta_1^{\pm}k_1^{\pm} - e_{33}^{\pm}\gamma_1^{\pm}k_1^{\pm},\\
n_2^{\pm} &= c_{13}^{\pm}\alpha_2^{\pm} - c_{33}^{\pm}(\beta_2^{\pm}\delta^{\pm} - \beta_3^{\pm}\omega^{\pm}) - e_{33}^{\pm}(\gamma_2^{\pm}\delta^{\pm} - \gamma_3^{\pm}\omega^{\pm}),
\end{aligned}
$$

$$
\begin{aligned}
n_3^{\pm} &= c_{13}^{\pm}\alpha_3^{\pm} - c_{33}^{\pm}(\beta_2^{\pm}\omega^{\pm} + \beta_3^{\pm}\delta^{\pm}) - e_{33}^{\pm}(\gamma_2^{\pm}\omega^{\pm} + \gamma_3^{\pm}\delta),\\
s_1^{\pm} &= e_{31}^{\pm}\alpha_1^{\pm} - e_{33}^{\pm}\beta_1^{\pm}k_1^{\pm} + \varepsilon_{33}^{\pm}\gamma_1^{\pm}k_1^{\pm},\\
s_2^{\pm} &= e_{31}^{\pm}\alpha_2^{\pm} - e_{33}^{\pm}(\beta_2^{\pm}\delta^{\pm} - \beta_3^{\pm}\omega^{\pm}) + \varepsilon_{33}^{\pm}(\gamma_2^{\pm}\delta^{\pm} - \gamma_3^{\pm}\omega^{\pm}),\\
s_3^{\pm} &= e_{31}^{\pm}\alpha_3^{\pm} - e_{33}^{\pm}(\beta_2^{\pm}\omega^{\pm} + \beta_3^{\pm}\delta^{\pm}) + \varepsilon_{33}^{\pm}(\gamma_2^{\pm}\omega^{\pm} + \gamma_3^{\pm}\delta^{\pm}),\\
\Delta_1 &= m_1^-(n_2^-s_3^- - n_3^-s_2^-) + m_2^-(n_3^-s_1^- - n_1^-s_3^-) + m_3^-(n_1^-s_2^- - n_2^-s_1^-)\\
p_{11} &= (s_2^-n_3^- - s_3^-n_2^-)/\Delta_1, \quad p_{12} = (m_2^-s_3^- - m_3^-s_2^-)/\Delta_1,\\
p_{13} &= (n_2^-m_3^- - n_3^-m_2^-)/\Delta_1, \quad p_{21} = (n_1^-s_3^- - n_3^-s_1^-)/\Delta_1,\\
p_{22} &= (s_1^-m_3^- - s_3^-m_1^-)/\Delta_1, \quad p_{23} = (n_3^-m_1^- - n_1^-m_3^-)/\Delta_1,\\
p_{31} &= (s_2^-n_1^- - s_1^-n_2^-)/\Delta_1, \quad p_{32} = (m_2^-s_1^- - m_1^-s_2^-)/\Delta_1,\\
p_{33} &= (m_1^-n_2^- - m_2^-n_1^-)/\Delta_1,\\
d_{11} &= p_{11}m_1^+ - p_{12}n_1^+ - p_{13}s_1^+, \quad d_{12} = p_{11}m_2^+ + p_{12}n_2^+ - p_{13}s_2^+,\\
d_{13} &= -p_{11}m_3^+ + p_{12}n_3^+ + p_{13}s_3^+, \quad d_{21} = p_{21}m_1^+ - p_{22}n_1^+ - p_{23}s_1^+,\\
d_{22} &= p_{21}m_2^+ - p_{22}n_2^+ - p_{23}s_2^+, \quad d_{23} = -p_{21}m_3^+ + p_{22}n_3^+ + p_{23}s_3^+,\\
d_{31} &= p_{31}m_1^+ - p_{32}n_1^+ - p_{33}s_1^+, \quad d_{32} = p_{31}m_2^+ - p_{32}n_2^+ - p_{33}s_2^+,\\
d_{33} &= -p_{31}m_3^+ + p_{32}n_3^+ + p_{33}s_3^+,\\
\rho_1 &= \alpha_1^+ - \alpha_1^-d_{11} - \alpha_2^-d_{21} + \alpha_3^-d_{31},\\
\rho_2 &= \alpha_2^+ - \alpha_1^-d_{12} - \alpha_2^-d_{22} + \alpha_3^-d_{32},\\
\rho_3 &= \alpha_3^+ + \alpha_1^-d_{13} + \alpha_2^-d_{23} - \alpha_3^-d_{33},\\
\xi_1 &= -\beta_1^+ - \beta_1^-d_{11} - \beta_2^-d_{21} + \beta_3^-d_{31},\\
\xi_2 &= -\beta_2^+ - \beta_1^-d_{12} - \beta_2^-d_{22} + \beta_3^-d_{32},\\
\xi_3 &= -\beta_3^+ + \beta_1^-d_{13} + \beta_2^-d_{23} - \beta_3^-d_{33},\\
q_1 &= -\gamma_1^+ - \gamma_1^-d_{11} - \gamma_2^-d_{21} + \gamma_3^-d_{31},
\end{aligned}
$$

$$q_2 = -\gamma_2^+ - \gamma_1^- d_{12} - \gamma_2^- d_{22} + \gamma_3^- d_{32},$$

$$q_3 = -\gamma_3^+ + \gamma_1^- d_{13} + \gamma_2^- d_{23} - \gamma_3^- d_{33},$$

$$\Delta_2 = q_1(\xi_2 m_3^+ - \xi_3 m_2^+) + q_2(\xi_3 m_1^+ - \xi_1 m_3^+) + q_3(\xi_1 m_2^+ - \xi_2 m_1^+),$$

$$a_{11} = (\xi_3 q_2 - \xi_2 q_3)/\Delta_2, \quad a_{12} = (q_3 m_2^+ - q_2 m_3^+)/\Delta_2,$$

$$a_{13} = (\xi_2 m_3^+ - \xi_3 m_2^+)/\Delta_2, \quad a_{21} = (\xi_1 q_3 - \xi_3 q_1)/\Delta_2,$$

$$a_{22} = (q_1 m_3^+ - q_3 m_1^+)/\Delta_2, \quad a_{23} = (\xi_3 m_1^+ - \xi_1 m_3^+)/\Delta_2,$$

$$a_{31} = (\xi_1 q_2 - \xi_2 q_1)/\Delta_2, \quad a_{32} = (q_1 m_2^+ - q_2 m_1^+)/\Delta_2,$$

$$a_{33} = (\xi_2 m_1^+ - \xi_1 m_2^+)/\Delta_2,$$

$$\eta_{11} = \rho_1 a_{11} + \rho_2 a_{21} - \rho_3 a_{31}, \quad \eta_{12} = \rho_1 a_{12} + \rho_2 a_{22} - \rho_3 a_{32},$$

$$\eta_{13} = \rho_1 a_{13} + \rho_2 a_{23} - \rho_3 a_{33}, \quad \eta_{21} = n_1^+ a_{11} + n_2^+ a_{21} - n_3^+ a_{31},$$

$$\eta_{22} = n_1^+ a_{12} + n_2^+ a_{22} - n_3^+ a_{32}, \quad \eta_{23} = n_1^+ a_{13} + n_2^+ a_{23} - n_3^+ a_{33},$$

$$\eta_{31} = s_1^+ a_{11} + s_2^+ a_{21} - s_3^+ a_{31}, \quad \eta_{32} = s_1^+ a_{12} + s_2^+ a_{22} - s_3^+ a_{32},$$

$$\eta_{33} = s_1^+ a_{13} + s_2^+ a_{23} - s_3^+ a_{33}.$$

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