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Damage standard models with a fixed convex

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The formal structure of a large class of standard damage models is being revisited. Particular attention is paid to the dependence of damage convex on the damage (internal) variables themselves. This dependence complicates the discussion about the existence and uniqueness conditions since the rate of internal variable is no longer a solution of a symmetric variational inequality. The alternative formulation is put forward involving a complementary set of internal variables. This method, leading to a fixed damage convex, brings back the rate response to an incremental behavior relevant to the generalized standard formalism. Some non-fixed convex models are replaced in the framework of damage models with fixed convex.

1. Introduction

Generalized standard models have been introduced in the scientific community thirty years ago by Halphen and Nguyen [1]. Their paper synthesizes and formalizes earlier work done by (among others) Ziegler [2] and Moreau [3], who brought out the role of convexity in the modelling of dissipative mechanisms. This concept has been rapidly adopted, thanks to its proper basis and its remarkable mathematical properties. In the early 80's the evolution of generalized standard structures was characterized by a general differential inclusion (see NGUYEN [4]):

$$\dot{\alpha} \in \mathcal{N}_{\mathcal{C}}(A), \tag{1.1}$$

where α is the set of internal variables, A the set of corresponding thermodynamic forces and $\mathcal{C}$ the convex domain of admissible thermodynamic forces. In general, the convex domain $\mathcal{C}$ depends on α. When it does not, it is relatively easy to determine the rate response: if $\lambda(t)$ is the parameter controlling the loading and $F(\alpha, \lambda)$ the expression of the total energy of the system at equilibrium, the rate $\dot{\alpha}$, associated with the loading rate $\dot{\lambda}$, is the solution of the following symmetric variational inequality:

$$\begin{aligned} &\dot{\alpha} \in \mathcal{N}_{\mathcal{C}}(A), \\ &\forall \delta\alpha \in \mathcal{N}_{\mathcal{C}}(A),\ (\delta\alpha - \dot{\alpha}).F''_{\alpha^2}(\alpha, \lambda).\dot{\alpha} + (\delta\alpha - \dot{\alpha}).F''_{\alpha\lambda}(\alpha, \lambda).\dot{\alpha} \geq 0. \end{aligned} \tag{1.2}$$

Thanks to this symmetry, the study of the variational inequality (1.2) does not raise particular difficulties: the classical results of convex analysis allow to obtain the conditions of existence and uniqueness of the solution $\dot{\alpha}$ (see NGUYEN [5]).

However, most of the damage models are built on a convex of admissible forces depending on the current state of internal variables. From usual and simple models using a single scalar variable (MARIGO [6], e.g.) to more complex ones dealing with anisotropic damage (CHABOCHE [7], HALM AND DRAGON [8], among others), with different damage mechanisms (CHABOCHE and MAIRE [9]) or coupling damage and plasticity (JU [10]), the damage variable plays two different roles: (i) it causes elastic properties alteration, (ii) it is responsible for hardening (the term "hardening" is used in this paper by analogy with plasticity), since it is intervening explicitly in the expression of reversibility threshold.

This latter dependence complicates the characterization of the rate response: $\dot{\alpha}$ becomes the solution of the following variational inequality:

$$\begin{aligned} &\dot{\alpha} \in \mathcal{N}_{\mathcal{C}(\alpha)}(A), \\ &\forall \delta\alpha \in \mathcal{N}_{\mathcal{C}(\alpha)}(A), \\ &(\delta\alpha - \dot{\alpha}).F''_{\alpha^2}(\alpha, \lambda).\dot{\alpha} + (\delta\alpha - \dot{\alpha}).F''_{\alpha\lambda}(\alpha, \lambda).\dot{\lambda}, \\ &\qquad + \dot{\alpha}.\left(D'_{\alpha}(\delta\alpha, \alpha) - D'_{\alpha}(\dot{\alpha}, \alpha)\right) \geq 0, \end{aligned} \tag{1.3}$$

where D denotes the dissipation potential. The study of (1.3) is more difficult: (i) The expression of D has to be known explicitly; in most cases, only the convex domain $\mathcal{C}$ is given, (ii) Inequality (1.3) is symmetric if and only if the term $\delta\alpha.D'_\alpha(\dot{\alpha},\alpha)$ is symmetric, i.e. $\delta\alpha.D'_\alpha(\dot{\alpha},\alpha) = \dot{\alpha}.D'_\alpha(\delta\alpha,\alpha)$, $\forall\delta\alpha \in \mathcal{N}_{\mathcal{C}(\alpha)}(A)$. If this latter requirement is not fulfilled, the variational inequality (1.3) admits no associated extremum principle and the discussion about the existence and uniqueness of the solution becomes cumbersome (NGUYEN [11]).

The paper comprises three parts:

(i) The first part aims at building a class of damage models in a rigorous standard framework (with a fixed convex). This class stems from a proper thermodynamic analysis while modifying the expression of the free energy: the set of internal variables is completed by a family of new *complementary* internal variables, so that the evolution laws are given by a differential inclusion with a fixed convex (1.1).

(ii) The addition of complementary variables can be considered as a method to recast classical models in the genuine standard framework. The paper gives the example of the Marigo model through two variant versions: the first one leads to obtain an identical mechanical response whereas the second one is fully equivalent to the original model (same free energy, same dissipation) while remaining standard with a fixed convex domain.

(iii) Adding terms to the free energy proves to be a fruitful source of damage models equivalent to the non-standard ones. Some examples are given; in particular, a model involving isotropic and kinematic hardening exhibits hereditary effects while remaining local in time. Another example allows to put back in a standard framework the widely used models whose damage variable is a tensor (say, d) and whose damage threshold involves the "tr d" terms.

2. Complementary internal variables and normal dissipation

Inviscid damage is the unique dissipative mechanism considered in this paper. It is quantified by scalar and/or tensorial internal variables, namely d_1, $d_2,\ldots,$ d_n. Strain is the observable variable. Usually, the set d_1, $d_2,\ldots,$ d_n intervenes not only in the elastic reversible energy but also in the damage threshold. The proposed method consists in adding a set of specific (scalar and/or tensorial) internal variables δ_1, $\delta_2,\ldots,$ δ_m. These sets are collectively denoted $d = (d_1,$ $d_2,\ldots,$ $d_n)$ and $\delta = (\delta_1,$ $\delta_2,\ldots,$ $\delta_m)$.

Let $w(\varepsilon, d, \delta)$ be the free energy per unit volume, considered as the thermodynamic potential. The two roles played in classical models by a single variable are here uncoupled: the sole variable d acts on the stiffness (or more generally, on the elastic behaviour) of the material, whereas the variable δ is responsible for hardening; the specific free energy thus splits into two terms:

$$w(\varepsilon, d, \delta) = \Phi \text{o} \varepsilon, d) + \psi(d, \delta). \tag{2.1}$$

However, for the sake of simplicity, the "blocked" free energy ψ is supposed to depend only on hardening variables:

$$w(\varepsilon, d, \delta) = \Phi(\varepsilon, d) + \psi(\delta). \tag{2.2}$$

The conjugate thermodynamic forces are given by:

$$Y(\varepsilon, d) = -\frac{\partial \Phi}{\partial d}(\varepsilon, d); \qquad Z(\delta) = \psi'(\delta). \tag{2.3}$$

Note that the force Y does not depend on δ. The convex domain $\mathcal{C}$ of admissible thermodynamic forces is defined by q inequalities: $\mathcal{F}_p(Y, Z) \leq 0\,(1 \leq p \leq q)$ where each $\mathcal{F}_p$ is a regular convex function such that $\mathcal{F}_p(0,0) \leq 0$ and the $\mathcal{F}'_p$ – vectors are independent.

The evolution laws are given by the maximum dissipation principle corresponding to the classical normality rule:

$$(\dot{d}, \dot{\delta}) \in \mathcal{N}_{\mathcal{C}}(Y, Z), \tag{2.4}$$

where $\mathcal{N}_{\mathcal{C}}(Y, Z)$ is the cone of the outer normals to $\mathcal{C}$ at the point (Y, Z). Thus, the variable d no longer acts as a hardening parameter since the force Z now plays this role; the domain of admissible thermodynamic forces, or reversibility domain, does not depend on the set (d, δ) of internal variables.

Table 1 synthesizes and compares the free energy, the thermodynamic forces, the reversibility domain and the evolution laws for both frameworks (classical and with complementary variables):

Table 1. Comparison between the usual models and the framework involving complementary variables.

	Usual framework	Addition of complementary variables
Internal variables	d	d, δ
Free energy	$\Phi(\varepsilon, d)$	$w(\varepsilon, d, \delta) = \Phi\varepsilon, d) + \psi(\delta)$
Thermodynamic forces	$Y = -\dfrac{\partial \Phi}{\partial d}(\varepsilon, d)$	$Y = -\dfrac{\partial \Phi}{\partial d}(\varepsilon, d); \quad Z = \psi'(\delta)$
Reversibility domain	$\mathcal{C}(d)$	$\mathcal{C}$ independent of (d, δ)
Evolution laws	$\dot{d} \in \mathcal{N}_{\mathcal{C}(d)}(Y)$	$(\dot{d}, \dot{\delta}) \in \mathcal{N}_{\mathcal{C}}(Y, Z)$

REMARK 1. CORDEBOIS and SIDOROFF [12] (or more recently HAYAKAWA *et al.* [13]) investigated the case of anisotropic materials undergoing plasticity and damage. A damage model is proposed by some affinity with classical plasticity: beside a damage variable affecting the elastic properties, a scalar variable, called β, is assumed to play for damage the role of a cumulative plastic strain in the

Prandtl–Reuss model, so that the damage hardening effects are controlled by the thermodynamic force conjugate to β and not by the internal variables themselves. On the one hand, the formalism presented in this section generalizes the work by Cordebois and Sidoroff and, on the other hand, as it will be shown in the next section, it is a way to retrieve standard models without modifying their mechanical response.

3. Two variant versions of a classical damage model

3.1. Reference damage model

The previous section has provided a general method to fix the reversibility domain of damage models by adding complementary variables responsible for hardening. As an illustration, consider the reference damage model by MARIGO [6] (called in the sequel "reference model"), whose key features are summarized below:

• Internal variable:

The damage variable, d, is scalar: it represents, for example, a density of microcracks whose orientation is random.

• Thermodynamic potential (specific free energy):

The free energy per unit volume $\Phi(\varepsilon, d)$ is assumed to be a homogeneous quadratic expression in terms of ε and linear in d:

$$\Phi(\varepsilon, d) = \frac{1}{2}(1-d)\varepsilon : C^0 : \varepsilon, \tag{3.1}$$

where C^0 is the initial stiffness tensor of the undamaged material.

• State laws (stress and thermodynamic force conjugate to damage):

$$\sigma = \frac{\partial \Phi}{\partial \varepsilon} = (1-d)C^0 : \varepsilon; \qquad Y = -\frac{\partial \Phi}{\partial d} = \frac{1}{2}\varepsilon : C^0 : \varepsilon. \tag{3.2}$$

The damage driving force Y can be seen as a damage energy release rate and is the proper quantity to enter the expression of the reversibility domain.

• Reversibility domain:

$$\mathcal{C}(d) = \{Y / Y - k(d) \leq 0\}, \tag{3.3}$$

where $k(d)$ is a positive function and represents the value to be reached by Y to start damage evolution. The function $k(d)$ and then the convex domain of admissible thermodynamic forces explicitly $\mathcal{C}(d)$ depend on the variable d.

• Evolution law:

The normal dissipative rule is supposed to govern the time-independent evolution of damage:

$$\dot{d} \in \mathcal{N}_{\mathcal{C}(d)}(Y). \tag{3.4}$$

3.2. First variant version with fixed convex

The purpose is to obtain a model in view of two constraints: (i) the reversibility convex domain has to be independent of d, (ii) the evolutions of the stress σ, the force Y and the damage variable d have to be the same as those given by the reference model, for a given deformation history.

Consider the following modified specific free energy:

$$w(\varepsilon, d, \delta) = \Phi(\varepsilon, d) + K(\delta) - k_0\delta = \frac{1}{2}(1-d)\varepsilon : C^0 : \varepsilon + K(\delta) - k_0\delta \tag{3.5}$$

where δ is a complementary scalar internal variable, $k_0 = k(0)$ a positive constant and $K(\delta)$ is the function defined by:

$$K(\delta) = \int_0^\delta k(x)dx. \tag{3.6}$$

The stress and thermodynamic forces conjugate to the internal variables d and δ are:

$$\begin{aligned} \sigma &= \frac{\partial w}{\partial \varepsilon}(\varepsilon, d, \delta) = \frac{\partial \Phi}{\partial \varepsilon}(\varepsilon, d) = (1-d)C^0 : \varepsilon, \\ Y &= -\frac{\partial w}{\partial d}(\varepsilon, d, \delta) = -\frac{\partial \Phi}{\partial d}(\varepsilon, d) = \frac{1}{2}\varepsilon : C^0 : \varepsilon, \\ Z &= -\frac{\partial w}{\partial \delta}(\varepsilon, d, \delta) = -k(\delta) + k_0. \end{aligned} \tag{3.7}$$

The convex domain $\mathcal{C}$ of the admissible thermodynamic forces takes the following form:

$$\mathcal{C} = \{(\hat{Y}, \hat{Z}) / G(\hat{Y}, \hat{Z}) = \hat{Y} + \hat{Z} - k_0 \le 0\}. \tag{3.8}$$

The evolution of both internal variables follows the normality rule $(\dot{d}, \dot{\delta}) \in \mathcal{N}_{\mathcal{C}}(Y, Z)$. This condition is equivalent to:

$$\dot{d} = \dot{\delta}. \tag{3.9}$$

Initially, the material is considered to be undamaged ($d = 0$, $\delta = 0$). From expression (3.9), it is straightforward to verify that, for a given strain ε, the mechanical response (σ, Y, d) is identical to that given by the reference model.

Remark 2. The reversibility convex has been fixed by adding a complementary variable. Another method to achieve this goal consists in choosing an appropriate damage variable. MARIGO [14] proves that replacing d by $\int\limits_0^d \frac{k(x)}{k_0} dx$ also leads to a fixed convex.

REMARK 3. In the framework of localization study, LORENTZ and ANDRIEUX [15], need the damage threshold independence with respect to the internal variable in order to be able to apply proper regularization methods. This is achieved by adding similarly a term in the free energy involving either the damage variable or a supplementary one. The additional term is interpreted as a kind of stored energy. In the previous paragraph, as well as in LORENTZ and ANDRIEUX [15], the modified model cannot be considered as fully equivalent to the original one since the free energy is changed by a supplementary term. The next part proves that a more complex choice for the additional term allows to retrieve the initial model.

3.3. Second variant version: thermodynamically equivalent model

The same formalism allows to obtain a model with a fixed convex reversible domain and keeping the free energy and the dissipation equal to the reference ones, which is not ensured in the previous variant version.

Let $K(d)$ and $H(d)$ be the following positive and strictly increasing functions:

$$K(d) = \int\limits_0^d k(\alpha) d\alpha; \qquad H(d) = 2 \int\limits_0^d \sqrt{k(\alpha)} d\alpha. \tag{3.10}$$

A complementary internal variable δ (initially null) is added; the specific free energy and the reversibility convex are chosen as follows:

$$\begin{gathered} w(\varepsilon, d, \delta) = \Phi(\varepsilon, d) + 2K(d) - 2K(H^{-1}(\delta)); \\ \mathcal{C} = \{(Y, Z) \in R^2 / Y + Z^2 \leq 0\}. \end{gathered} \tag{3.11}$$

The thermodynamic forces are then:

$$Y = \frac{1}{2} \varepsilon : C^0 : \varepsilon - 2k(d); \qquad Z = \sqrt{k(H^{-1}(\delta))}. \tag{3.12}$$

The normality rule gives:

$$\dot{d} = \lambda \quad \text{and} \quad \dot{\delta} = 2\lambda Z, \tag{3.13}$$

where $\lambda \geq 0$ is the damage multiplier. One infers the following differential equation:

$$\frac{\dot{\delta}}{2\sqrt{k(H^{-1}(\delta))}} = \dot{d}. \tag{3.14}$$

Recalling the expression for the derivative $(H^{-1})'(\delta)$:

$$(H^{-1})'(\delta) = \frac{1}{H'(H^{-1}(\delta))} = \frac{1}{2\sqrt{k(H^{-1}(\delta))}} \tag{3.15}$$

and integrating Equation (3.14) leads to the relation between δ and d:

$$\delta = H(d). \tag{3.16}$$

Substituting δ for $H(d)$ into (3.11) allows to obtain the free energy and the reversibility domain of the reference model. Consequently, free and dissipated energy are equal and the two models are strictly equivalent from the thermodynamic point of view.

4. Some general examples of damage models with fixed convex

The method exposed in Sec. 3 can be seen not only as a way to reformulate classical models but also as a method to generate a new class of damage models. In this section, two examples are given: a simple model combining 'isotropic' and 'kinematic' hardening and a model whose hardening threshold involves the damage variable.

4.1. A damage model combining isotropic and kinematic hardening

The variable d is assumed to be a n-uple. Consider $n+1$ complementary internal variables $\delta = (\beta, \gamma)$ with $\beta = (\delta_1, \dots, \delta_n)$, $\delta_{n+1} = \gamma$. A simple expression is adopted for the specific free energy:

$$w(\varepsilon, d, \delta) = \Phi(\varepsilon, d) + \psi(\beta, \gamma), \qquad \psi(\beta, \gamma) = \varphi(\beta) + \chi(\gamma). \tag{4.1}$$

The thermodynamic forces conjugate to d and δ are respectively Y and $Z = (X, \Gamma)$:

$$Y = -\frac{\partial \Phi}{\partial d}(\varepsilon, d); \qquad X = -\varphi'(\beta), \qquad \Gamma = -\chi'(\gamma). \tag{4.2}$$

These forces are constrained to stay inside the convex domain $\mathcal{C}$ independent of the internal variables; $\mathcal{C}$ is defined by:

$$G(Y, Z) = \sqrt{(Y+X).(Y+X)} + \Gamma - K_0 \leq 0 \tag{4.3}$$

with K_0 being a positive constant. The evolution laws are governed by the normality rule $(\dot{d}, \dot{\delta}) \in \mathcal{N}_{\mathcal{C}}(Y, Z)$ and account for isotropic and kinematic hardening effects.

In order to compare this model with usual ones, the forces X and Γ are expressed as functions of Y and d by elimination of the internal variables:

$$X(d) = -\varphi'(d); \qquad \Gamma(t) = -\chi'\left[\int_0^t \frac{\dot{d}.(Y-\varphi'(d))}{\sqrt{(Y-\varphi'(d)).(Y-\varphi'(d))}}d\tau\right]. \tag{4.4}$$

Substituting (4.4) into (4.3) leads to:

$$g(Y,d) = \sqrt{(Y-\varphi'(d)).(Y-\varphi'(d))} - \left(K_0 + \chi'\left[\int_0^t \frac{\dot{d}.(Y-\varphi'(d))}{\sqrt{(Y-\varphi'(d)).(Y-\varphi'(d))}}d\tau\right]\right) \leq 0. \tag{4.5}$$

The evolution of d is given by:

$$\dot{d} = \begin{cases} \lambda\dfrac{Y-\varphi'(d)}{\sqrt{(Y-\varphi'(d)).(Y-\varphi'(d))}} & \text{if} \quad g(Y,d)=0 \quad \text{with} \quad \lambda \geq 0, \\ 0 & \text{if} \quad g(Y,d) < 0. \end{cases} \tag{4.6}$$

The comparison between $G(Y,Z)=0$ and $g(Y,d)=0$ clearly shows that the model (4.1)–(4.3) exhibits hereditary effects and involves the whole history of d, Y and $\dot{d}$, whereas it remains local in time (in the sense that only the instantaneous values of the forces intervene in G).

REMARK 4. In the case of isotropic hardening ($\varphi = 0$) and when the vector Y keeps a fixed direction whose unit vector is $\tilde{Y}$, the criterion $g(Y,d)=0$ becomes local in time again.

REMARK 5. The model (4.5)–(4.6) is not a generalized standard one. Thus, models apparently not generalized standard and handling complex hereditary effects can be equivalently reduced to more convenient and standard expressions.

4.2. Damage threshold with a "tr d" term

In this second example, the set of n internal variables is, once more, denoted by $d = (d_1, \ldots, d_n)$. The expression of the thermodynamic force $Y = (Y_1, \ldots, Y_n)$ is obtained by differentiation of the specific free energy $\Phi(\varepsilon, d)$:

$$Y_i = -\frac{\partial \Phi}{\partial d_i}(\varepsilon, d). \tag{4.7}$$

These thermodynamic forces are constrained to stay inside the convex reversibility domain $\mathcal{C}(d)$ depending on the present state of the internal variables:

$$\begin{gathered} \mathcal{C}(d) = \{Y / g(Y_1, \ldots, Y_n, d_1, \ldots, d_m) \leq 0\} \\ \text{with} \quad g(Y_1, \ldots, Y_n, d_1, \ldots, d_m) = \operatorname*{Max}_{1\leq i \leq n} Y_i - a(d_1 + \ldots + d_m) - k_0. \end{gathered} \tag{4.8}$$

The evolution of d is classically given by the normality rule $\dot{d} \in \mathcal{N}_{\mathcal{C}(d)}(Y)$. Although the damage threshold explicitly depends on the damage variable d, it can be proved that this model matches a version with a fixed convex by considering the set of complementary variables $\delta = (\delta_1, \ldots, \delta_m)$ and the specific free energy:

$$w(\varepsilon, d, \delta) = \Phi(\varepsilon, d) + \frac{1}{2}\frac{a}{m}(\delta_1^2 + \ldots + \delta_m^2). \tag{4.9}$$

The thermodynamic force $Z_j (1 \leq j \leq m)$, defined by (4.10), corresponds to each scalar internal variable δ_j:

$$Z_j = -\frac{a}{m}\delta_j. \tag{4.10}$$

When expressed in the space of thermodynamic forces, the damage criterion $g = 0$ becomes:

$$G(Y_1, \ldots, Y_n, Z_1, \ldots, Z_m) = \underset{1 \leq i \leq n}{\mathrm{Max}}\, Y_i + (Z_1 + \ldots + Z_m) - k_0 = 0 \tag{4.11}$$

and the evolution of the internal variables (d, δ) is normal to the convex $\mathcal{C}$ defined by $G \leq 0$, i.e. $(\dot{d}, \dot{\delta}) \in \mathcal{N}_{\mathcal{C}}(Y, Z)$. This latter hypothesis leads to $Z_1 + \ldots + Z_m = a(d_1 + \ldots + d_m)$ and so the equivalence between the criteria $g = 0$ and $G = 0$ is proved.

Remark 6. It can be noted that the general class of models (4.7)–(4.8) includes the particular case of damage thresholds in which intervenes the trace of a second order tensorial damage variable: many works in the literature treat anisotropic damage with second order tensors whose trace is commonly the scalar parameter governing hardening effects (see, e.g. Chaboche and Maire [9], Halm and Dragon [8] or Besson *et al.* [16] for a more general review).

5. Conclusion

A general method has been proposed to model dissipative mechanisms, such as damage, in the framework of Thermodynamics of Irreversible Processes and, in particular, of generalized standard approach with a fixed convex. The key point consists in substituting the internal variables intervening in the threshold by a set of thermodynamic forces depending on complementary internal variables. Consequently, the results obtained and tested for many years for the models with a fixed convex (such as plasticity with hardening) can be directly applied to a general class of damage models; this could not be done simply before. In particular, as pointed out in the Introduction, the bifurcation analysis becomes much easier. Another major contribution of the method exposed in this paper consists in its capacity to propose some classes of damage models with a fixed convex. Thus a non-standard model involving hereditary effects has been found to be equivalent to a standard model local in time.

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