Brief Note

Existence of zero-group velocity modes in an incompressible plate

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Zero-group velocity (ZGV) Lamb modes are studied in an incompressible orthotropic plate. The existence of such modes critically depends on the anisotropy parameter $a = \frac{(c_{11} + c_{22} - 2c_{12} - 4c_{66})}{c_{66}}$. With materials having $a > -1$, none of the modes possesses any ZGV points and every mode has such a point if $a < -1$. Several modes have multiple ZGV points.

Key words: Lamb modes, anisotropy, zero-group velocity points.

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1. Introduction

Lamb waves are an important tool for non-destructive testing of materials [1]. This is done by studying the shape of Lamb modes generated by a perturbation of the surface of a mechanical structure, see for example [2]. In a plate, free of defects, the phase speed of each mode, as a function of wave number, exhibits a plateau region and asymptotically approaches a fixed speed [3]. However in the frequency-wave number spectrum, certain modes are anomalous in the sense that, over a narrow frequency interval, the group velocity is directed opposite to the phase velocity. The group velocity vanishes at a point where the dispersion curve undergoes a change in the sign of its slope. Such a point is called a zero-group velocity (ZGV) point. The existence of a ZGV mode was first pointed out by Tolstoy and Usdin [4] in 1957 and these modes have been investigated theoretically and experimentally by several authors [5–12]. Recently Hussain and Ahmad [13] considered ZGV points in the spectrum of Lamb modes in a compressible orthotropic plate. It was found that, in addition to modes with a single ZGV point, a large number of modes exist with multiple such points.

Several problems in elasticity become relatively easy when the material is assumed to be incompressible. This happens because, in this case, a displacement
potential function can be introduced [14, 15]. This assumption can be justified in
case of materials such as Kraton polymer, thermoplastic elastomers and rubber
composites [16]. Rogerson [17] and Ogden and Roxburgh [18] have discussed
the dispersion relation for incompressible elastic plate but they did not consider
ZGV Lamb modes.

In the present paper, we will consider the effect of the incompressibility con-
straint on the ZGV modes. Let us define a parameter of elastic anisotropy

\begin{equation}
(1.1) \quad a = \frac{c_{11} + c_{22} - 2c_{12} - 4c_{66}}{c_{66}}.
\end{equation}

This parameter satisfies the constraint

\begin{equation}
(1.2) \quad a \geq -4.
\end{equation}

We shall consider symmetric Lamb modes in an incompressible orthotropic
plate and show that the numerical value of the anisotropy parameter plays a crit-
ical role in the existence, or otherwise, of ZGV points in the spectrum of a ma-
terial. We shall establish that

(i) No ZGV mode exists if \( a > -1 \).

(ii) Every mode, with the possible exception of the lowest few modes, is a ZGV
mode if \( a < -1 \).

The above results are established analytically and are borne out by dispersion
curves. These curves also exhibit multiple ZGV points for materials with the
anisotropy parameter in the range \(-4 < a < -3\). Since \( a = 0 \) for isotropic
as well as transversely isotropic materials, it follows that the incompressibility
constraint will preclude anomalous modes in such materials.

2. ZGV Lamb modes

The symmetric dispersion relation for the waves propagating in incompress-
ible elastic plates, having thickness \( 2h \), is given by [17, 18]

\begin{equation}
(2.1) \quad \frac{\tan(s_1kh)}{\tan(s_2kh)} = \frac{s_1(1 - s_2^2)^2}{s_2(1 - s_1^2)^2},
\end{equation}

where \( s_1^2 \) and \( s_2^2 \) are roots of the following equation

\begin{equation}
(2.2) \quad s^4 + \left( a + 2 - \frac{\rho c^2}{c_{66}} \right) s^2 + \left( 1 - \frac{\rho c^2}{c_{66}} \right) = 0.
\end{equation}

If we denote dimensionless wave number \( kh \) by \( x \) and \( c/\sqrt{c_{66}/\rho} \) by \( y \), then
Eq. (2.1) becomes

\begin{equation}
(2.3) \quad \frac{\tan(s_1 x)}{\tan(s_2 x)} = \frac{s_1(1 - s_2^2)^2}{s_2(1 - s_1^2)^2},
\end{equation}
with \( s_1 \) and \( s_2 \) given by

\[
\begin{align*}
  s_1 &= \sqrt{-\frac{(a + 2 - y^2) - \sqrt{(a + 2 - y^2)^2 - 4(1 - y^2)}}{2}}, \\
  s_2 &= \sqrt{-\frac{(a + 2 - y^2) + \sqrt{(a + 2 - y^2)^2 - 4(1 - y^2)}}{2}}.
\end{align*}
\]

The ZGV modes manifest themselves in the \( \omega - c \) plane. Let \( c_T = \sqrt{c_{66}/\rho} \) denote the shear velocity of the wave and we define the dimensionless variable

\[
u = \frac{\omega h}{\sqrt{c_{66}/\rho}} = yx.
\]

In terms of \( u \) and \( y \) Eq. (2.3) becomes

\[
\tan(s_1 u/y) = \frac{s_1(1 - s_2^2)^2}{s_2(1 - s_1^2)^2}.
\]

Let \( S_n, n = 1, 2, \ldots \) denote the \( n \)-th mode in the spectrum. To check whether or not a mode is anomalous, we look at the slope of each mode for large as well as small \( y \).

In order to examine slope of the mode \( S_n \) we rewrite Eq. (2.7) in the form

\[
f(u, y) = \tan(s_1 u/y)s_2(1 - s_1^2)^2 - \tan(s_2 u/y)s_1(1 - s_2^2)^2 = 0.
\]

For large \( y \),

\[
\begin{align*}
  s_1 &\simeq i, \\
  s_2 &\simeq y \left(1 - \frac{a + 1}{2y^2}\right),
\end{align*}
\]

and Eq. (2.8) becomes

\[
\tan \left( u_n \left(1 - \frac{a + 1}{2y^2}\right) \right) \approx 0.
\]

We will determine the slope \( dy/du \) by using the following formula

\[
\frac{dy}{du} = -\frac{\partial f/\partial u}{\partial f/\partial y}.
\]

The partial derivatives can be approximated, for large \( y \), as follows

\[
\begin{align*}
  \frac{\partial f}{\partial u} &\approx i(4 - y^4), \\
  \frac{\partial f}{\partial y} &\approx -i u_n(a + 1)y.
\end{align*}
\]
Using these expressions we have, for large $y$,

\begin{equation}
\frac{dy}{du} \simeq -\frac{y^3}{u_n(a + 1)}, \quad n = 1, 2, \ldots.
\end{equation}

If $a + 1 > 0$, then

\begin{equation}
\frac{dy}{du} < 0 \quad \text{for } S_n, \ n \geq 1.
\end{equation}

For $a + 1 < 0$, we have

\begin{equation}
\frac{dy}{du} > 0 \quad \text{for } S_n, \ n \geq 1.
\end{equation}

Now we will estimate the slope when $y$ approaches the line $y = 1$ from above. First consider the case when $a + 1 < 0$. Let $b = -(a + 1) > 0$ and $y^2 = 1 + \epsilon^2$, where $\epsilon$ is a small positive number. Then

\begin{equation}
s_1^2 = \frac{b + \epsilon^2 - \sqrt{(b + \epsilon^2)^2 + 4\epsilon^2}}{2},
\end{equation}

\begin{equation}
s_1^2 \simeq \frac{1}{2} \left[ b + \epsilon^2 - b \sqrt{1 + \epsilon^2 \frac{2b + 4}{b^2}} \right],
\end{equation}

\begin{equation}
s_1^2 \simeq -\frac{\epsilon^2}{b},
\end{equation}

where we have ignored terms of order $\epsilon^3$ or higher. Also

\begin{equation}
s_2^2 \simeq b.
\end{equation}

With $s_1 = \frac{i\epsilon}{\sqrt{b}}$, $s_2 = \sqrt{b}$ Eq. (2.8) becomes

\begin{equation}
\tanh \left( \frac{\epsilon u}{\sqrt{b}} \right) = \frac{\epsilon(1 - b)^2}{b(1 + \epsilon^2 b^2)^2}.
\end{equation}

or

\begin{equation}
\tanh \left( \frac{\epsilon u}{\sqrt{b}} \right) \approx \frac{(1 - b)^2}{b} \tan(\sqrt{b}u)\epsilon.
\end{equation}

The partial derivatives can be approximated, in this case, as follows

\begin{equation}
\frac{\partial f}{\partial u} \simeq -i\epsilon \sqrt{b} \left( \frac{(1-b)^2}{\sqrt{b}} + \frac{u^2 \sqrt{b}}{(1-b)^2} - \frac{1}{\sqrt{b}} \right),
\end{equation}

\begin{equation}
\frac{\partial f}{\partial y} \simeq \frac{i\epsilon u}{b^2(1-b)^2} \left[ (1+b)(1-b)^2-(1-b)^4(1+b-b^2)+8b(1-b)-b(a+3)u^2 \right].
\end{equation}
Using these expressions we have, for $y \to 1^+$,

\[
\begin{align*}
\frac{dy}{du} &\approx \frac{b^3[(b-2)(1-b)^2+u_n^2]}{u_n(1+b)(1-b)^2-(1-b)^4(1+b-b^2)+8(1-b)-b(a+3)u_n^2}, \\
\end{align*}
\]

for $n = 1, 2, \ldots$. From Eq. (2.24), it is clear that when $a + 3 > 0$ and $y \to 1^+$, $dy/du < 0$ for sufficiently large $u_n$.

Next let $1 + a > 0$. In this case

\[
\begin{align*}
s_1^2 &\approx -(a + 1), \\
s_2^2 &\approx \frac{\epsilon^2}{a + 1}.
\end{align*}
\]

Equation (2.8) becomes

\[
\frac{\tanh(\sqrt{a+1}u)}{\tan(\sqrt{a+1}u)} \approx \frac{a + 1}{\epsilon} \frac{(1 - \frac{\epsilon^2}{a+1})^2}{(a + 2)^2}.
\]

The above equation was considered by Ahmad [3]. All modes, except $S_0$, approach the line $y = 1$ with negative slope.

Thus we have established that, in case of materials with $a + 1 > 0$, the modes maintain their negative slope from their inception when $y \to \infty$ to the eventual non-oscillation limit when $y \to 1^+$. These modes are free of anomalous behavior and do not possess any ZGV points.

However for materials with $a + 1 < 0$ and $a + 3 > 0$ i.e. $-3 < a < -1$, Eq. (2.15) indicates that the slope is positive for large $y$ while Eq. (2.24) shows that, for sufficiently large $u_n$, the slope becomes negative when $y \to 1^+$. This change in the sign of slope leads to at least one ZGV point for each mode in the spectrum, with the possible exception of the first few modes.

The case $-4 < a < -3$ still needs to be sorted out. For this purpose, we calculate the slope for that value of $y$, say $y_0$, at which the lowest mode, $S_0$, is born. To find $y_0$, let $u \to 0$ in Eq. (2.7). This leads to

\[
\begin{align*}
(1 - s_1^2)^2 &= (1 - s_2^2)^2, \\
\end{align*}
\]

or

\[
\begin{align*}
s_1^2 + s_2^2 &= 2,
\end{align*}
\]

is the only acceptable solution of Eq. (2.7) when $-4 < a < -3$. From Eq. (2.2),

\[
\begin{align*}
s_1^2 + s_2^2 &= -(a + 2 - y^2),
\end{align*}
\]

hence

\[
\begin{align*}
y_0^2 &= a + 4,
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]
For the range of \( a \) under consideration, \( 0 < y_0 < 1 \). The lowest mode starts above or below the line \( y = 1 \) according as \( a + 3 \) is positive or negative.

The derivative is given by the following expression:

\[
(2.33) \quad \left. \frac{dy}{du} \right|_{y=y_0} = \frac{4s_1^2(a + 4)s_2 \sin^2(s_2u_n/y_0)}{2u_n s_2(a + 2)\sqrt{a+4}\cos^2(s_2u_n/y_0) + (a + 4)(3a + 10)\sin(s_2u_n/y_0)},
\]

for \( n = 1, 2, \ldots \).

When \( y = y_0, s_1, s_2 \) respectively are found from Eqs (2.4) and (2.5) as

\[
s_1 = \sqrt{1 - y_0}, \quad s_2 = \sqrt{1 + y_0}.
\]

Since \( y_0 \) does not equal zero or 1, vanishing of \( \cos(s_2u_n/y_0) \) is ruled out by Eq. (2.7). Also \( a + 2 < 0 \), hence

\[
(2.34) \quad \left. \frac{dy}{du} \right|_{y=y_0} < 0,
\]

for sufficiently large \( u_n \). Thus modes will undergo a change of sign and each of them will possess a ZGV point.

We have established the result that, with the possible exception of first few modes, no mode will be anomalous if \( a + 1 > 0 \) and every mode will be anomalous if \( a + 1 < 0 \).

Table 1 shows values of anisotropy parameter \( a \) for a few orthotropic materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>Value of anisotropy parameter ( a )</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iodic acid</td>
<td>-0.4620</td>
<td>20</td>
</tr>
<tr>
<td>Barium sodium niobate</td>
<td>-0.3421</td>
<td>20</td>
</tr>
<tr>
<td>Lead chloride</td>
<td>4.5625</td>
<td>21</td>
</tr>
<tr>
<td>Gallium</td>
<td>-1.1166</td>
<td>21</td>
</tr>
<tr>
<td>Cadmium formate</td>
<td>-2.4894</td>
<td>21</td>
</tr>
<tr>
<td>Forsterite</td>
<td>0.8952</td>
<td>21</td>
</tr>
<tr>
<td>Rubidium sulfate</td>
<td>0.4043</td>
<td>21</td>
</tr>
<tr>
<td>Calcium formate</td>
<td>-3.1489</td>
<td>21</td>
</tr>
</tbody>
</table>
We have plotted the dispersion curves, using the technique of Honarvar et al. [19].

Figure 1 exhibits the dispersion curves for barium sodium niobate which has $a = -0.3421$.

![Figure 1](image1.png)

**Fig. 1.** Symmetric Lamb modes for an incompressible orthotropic barium sodium niobate material with $a = -0.3421$ in $(\omega, c)$ plane. No mode is anomalous.

Figure 2 depicts the anomalous modes $S_2, S_3, S_4, S_5$ for the spectrum of calcium formate which has $a = -3.1489$.

![Figure 2](image2.png)

**Fig. 2.** Symmetric Lamb modes for an incompressible orthotropic calcium formate material with $a = -3.1489$ in $(\omega, c)$ plane. All modes are anomalous except $S_0$ and $S_1$.

The Lamb modes, having three ZGV points, $S_6, S_7, S_8, S_9, S_{10}$ for calcium formate are plotted in Figure 3.
In Figure 3, we have plotted the first seven Lamb modes for calcium formate in the $(\omega, c)$ plane.

Fig. 3. Symmetric Lamb modes for an incompressible orthotropic calcium formate material with $\alpha = -3.1489$ in $(\omega, c)$ plane.

In Figure 4, we have plotted first seven Lamb modes for calcium formate in the $(k, \omega)$ plane.

Fig. 4. First 7 symmetric Lamb mode for an incompressible orthotropic calcium formate material with $\alpha = -3.1489$ in $(k, \omega)$ plane.

One can easily observe that all modes, except $S_0$ and $S_1$, have at least one ZGV point, but $S_6$ is the first mode which has three ZGV points. The modes $S_4$ and $S_5$ also seem to have multiple ZGV points but a closer inspection does not bear this out. This becomes clear in Fig. 5, where we have focused on that part of $S_4$ where the second and third ZGV points appear to exist. However a similar
Existence of zero-group velocity modes.

<table>
<thead>
<tr>
<th>Thickness/Wavelength k(h/π)</th>
<th>Frequency Thickness (h/π) MHz:mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.36</td>
<td>18.657</td>
</tr>
<tr>
<td>5.38</td>
<td>18.6571</td>
</tr>
<tr>
<td>5.4</td>
<td>18.6573</td>
</tr>
<tr>
<td>5.42</td>
<td>18.6574</td>
</tr>
<tr>
<td>5.44</td>
<td>18.6575</td>
</tr>
<tr>
<td>5.46</td>
<td>18.6576</td>
</tr>
</tbody>
</table>

**Fig. 5.** The $S_4$ symmetric Lamb mode for an incompressible orthotropic calcium formate material in $(k, \omega)$ plane.

<table>
<thead>
<tr>
<th>Thickness/Wavelength k(h/π)</th>
<th>Frequency Thickness (h/π) MHz:mm</th>
</tr>
</thead>
<tbody>
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<tr>
<td>13.485</td>
<td>4.085</td>
</tr>
<tr>
<td>13.49</td>
<td>4.09</td>
</tr>
</tbody>
</table>

**Fig. 6.** The $S_6$ symmetric Lamb mode with second and third ZGV points for an incompressible orthotropic calcium formate material in $(k, \omega)$ plane.

Focus on the corresponding domain of $S_6$ in Fig. 6 clearly indicates the existence of the second and third ZGV points.

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References


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