Neutrality of coated holes in the presence of screw dislocation dipoles or circular thermal inclusions

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In this paper, we consider the design of neutral coated holes in two particular cases when the thick coating itself is altered by the presence of some form of material imperfection. In the first case we consider anti-plane deformations of a linearly elastic solid when the thick coating applied to the hole includes a screw dislocation dipole. In the second case, we investigate the design of neutral coated holes in plane elasticity when the thick coating contains a circular thermal inclusion and the surrounding linearly elastic solid is subjected to uniform remote hydrostatic stresses. The design is achieved by constructing particular forms of the conformal mapping function for the coating itself. Several examples are presented to demonstrate the resulting solutions. Our numerical results show that the existence of the screw dislocation dipole or the circular thermal inclusion in the coating exerts a significant influence on the shape of the neutral coated hole.

Key words: neutral holes, thick coating, screw dislocation dipole, thermal inclusion, conformal mapping, anti-plane elasticity, plane elasticity.

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1. Introduction

A hole introduced into an elastic body will inevitably disturb the body’s original stress field and often lead to a stress concentration. MANSFIELD [1] was among the first to recognize the feasibility of designing a reinforced “neutral” hole which eliminates any stress concentrations introduced by the hole and hence does not disturb the original stress field in the uncut body. Relevant studies on the design of neutral holes and neutral inclusions in composite materials are abundant and can be found in [2–14]. The concept of neutral inclusions has also been adapted to the design of cloaking structures [15–17].
In the aforementioned discussions of neutrality in composites, the composites themselves are assumed to be free of the presence of dislocations. The existence of dislocations in solid materials, however, has now been confirmed experimentally [18]. Considerable efforts have been made to associate plastic flow in crystalline solids with motions of dislocations (see for example, [19–23]). It is therefore natural to ask whether it remains possible to design neutral holes or inclusions when dislocations are present in the composite? Furthermore, it has been established that the shape of a neutral hole in an isotropic field (or hydrostatic stress field) is certainly circular [7]. Will this interesting and important result persist if the thick coating reinforcing the hole surrounds a circular thermal inclusion?

In this paper, we indeed address each of the aforementioned questions. We begin with an investigation of the neutrality of coated holes in anti-plane elasticity when a screw dislocation dipole is present in the thick coating itself. Secondly, we address the neutrality of coated holes in plane elasticity when the thick coating contains a circular thermal inclusion and the solid is subjected to uniform remote hydrostatic stresses. Our method includes the introduction of particular forms of the corresponding conformal mapping which includes either a logarithmic function to account for the existence of the dislocation dipole or a first-order pole in the case of the circular thermal inclusion. These conformal mappings are then the key to the successful solution of the corresponding boundary value problems. We validate and illustrate our solutions via the use of several numerical examples.

2. Complex variable formulation

Under anti-plane shear deformations of a linearly isotropic elastic material, in a Cartesian coordinate system, the two shear stress components $\sigma_{31}$ and $\sigma_{32}$, the out-of-plane displacement $w$ and the stress function $\varphi$ can be expressed in terms of a single analytic function $f(z)$ of the complex variable $z = x_1 + ix_2$ as [24]

$$
\sigma_{32} + i\sigma_{31} = \mu f'(z), \quad \varphi + i\mu w = \mu f(z),
$$

where $\mu$ is the shear modulus and the two stress components can be expressed in terms of the same stress function as [24]

$$
\sigma_{32} = \varphi_1, \quad \sigma_{31} = -\varphi_2.
$$

For plane deformations of an isotropic elastic material, the stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$, displacements $(u_1, u_2)$ and stress functions $(\varphi_1, \varphi_2)$ can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + ix_2$ as [25]
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\[
\sigma_{11} + \sigma_{22} = 2[\phi'(z) + \bar{\phi}'(z)], \\
\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\phi''(z) + \psi'(z)],
\]

(2.3)

\[
2\mu(u_1 + iu_2) = \kappa\phi(z) - z\bar{\phi}'(z) - \psi(z), \\
\varphi_1 + i\varphi_2 = i[\phi(z) + z\bar{\phi}'(z) + \psi(z)],
\]

(2.4)

where \(\kappa = 3 - 4\nu\) for plane strain and \(\kappa = (3 - \nu)/(1 + \nu)\) for plane stress with \(\nu\) \((0 \leq \nu \leq 1/2)\) being the Poisson’s ratio. In addition, the stresses are related to the stress functions through [24]

\[
\sigma_{11} = -\varphi_{1,2}, \quad \sigma_{12} = \varphi_{1,1}, \\
\sigma_{21} = -\varphi_{2,2}, \quad \sigma_{22} = \varphi_{2,1}.
\]

(2.5)

3. Neutral coated holes in the presence of screw dislocation dipoles

Consider a domain in \(\mathbb{R}^2\), infinite in extent, containing a hole occupying a simply connected region \(\Omega\), as shown in Fig. 1. The hole is surrounded by a thick coating occupying a doubly connected region \(S_2\) and the coating is perfectly bonded to the surrounding matrix occupying the region \(S_1\). The coating-matrix interface is denoted by \(L_1\) whilst the traction-free boundary of the hole.

Fig. 1. A neutral coated hole in the presence of a screw dislocation dipole in the thick coating.
is denoted by $L_2$. The matrix is subjected to uniform remote anti-plane shear
stresses $(\sigma_{31}^\infty, \sigma_{32}^\infty)$ while the coating is under the action of a screw dislocation
dipole composed of a screw dislocation with the Burgers vector $b_3$ located at
$z = z_1$ and a second screw dislocation with the opposite Burgers vector $-b_3$
located at $z = z_2$ (see Fig. 1). Throughout the paper, the subscripts 1 and 2
are used to identify the respective quantities in $S_1$ and $S_2$. Our objective is to
design the shape of the thick coating to achieve neutrality, i.e.
to ensure that the
coated hole will not disturb the original uniform stress field in the surr
rounding
matrix. The incorporation of a dislocation dipole as opposed to a disloc
ation in
the coating is to ensure the single-valuedness of the displacement field on any
contour in the matrix surrounding the coating.

The boundary value problem for the two-phase composite takes the following
form:

\begin{align}
f_2(z) + \overline{f_2(z)} &= \Gamma f_1(z) + \overline{\Gamma f_1(z)}, \quad f_2(z) - \overline{f_2(z)} = f_1(z) - \overline{f_1(z)}, z \in L_1; \\
f_2(z) + \overline{f_2(z)} &= 0, \quad z \in L_2;
\end{align}

\begin{align}
f_2(z) &\approx \frac{b_3}{2\pi} \ln(z - z_1) + O(1), \quad z \rightarrow z_1, \\
f_2(z) &\approx -\frac{b_3}{2\pi} \ln(z - z_2) + O(1), \quad z \rightarrow z_2;
\end{align}

\begin{align}
f_1(z) &= Cz, \quad z \in S_1,
\end{align}

where $\Gamma = \mu_1/\mu_2$ and the complex number $C$ is given by

\begin{align}
C &= \frac{\sigma_{32}^\infty + i\sigma_{31}^\infty}{\mu_1}.
\end{align}

Consider the following conformal mapping function for the coating

\begin{align}
z = \omega(\xi) = q \ln \frac{\xi - \bar{\xi}_1^{-1}}{\xi - \bar{\xi}_2^{-1}} + \sum_{n=1}^{+\infty} (a_n \xi^n + a_{-n} \xi^{-n}), \quad \xi = \omega^{-1}(z), \quad r \leq |\xi| \leq 1,
\end{align}

where $q, a_n, a_{-n}, n = 1, 2, \ldots, +\infty$ are complex constants to be determined,
$\xi_1 = \omega^{-1}(z_1)$ and $\xi_2 = \omega^{-1}(z_2)$. Using the mapping function, the coating in
the $z$-plane is mapped onto an annulus $r \leq |\xi| \leq 1$ in the $\xi$-plane, the coating-
matrix interface $L_1$ in the $z$-plane is mapped onto $|\xi| = 1$ in the $\xi$-plane, the
traction-free boundary of the hole $L_2$ in the $z$-plane is mapped onto $|\xi| = r$ in
the $\xi$-plane, $z = z_1$ is mapped onto $\xi = \xi_1$ and $z = z_2$ is mapped onto $\xi = \xi_2$ (see Fig. 2). The appearance of the logarithmic function $\ln \frac{\xi - \xi_1^{-1}}{\xi - \xi_2^{-1}}$ in the mapping function is to account for the existence of the screw dislocation dipole.

Fig. 2. The problem in the $\xi$-plane.

By enforcing the continuity conditions of traction and displacement across the coating-matrix interface $L_1$ in Eq. (3.1a), we arrive at

\[(3.4)\quad f_2(\xi) = f_2(\omega(\xi)) = \frac{C(\Gamma + 1)}{2} \omega(\xi) + \frac{\bar{C}(\Gamma - 1)}{2} \bar{\omega} \left( \frac{1}{\xi} \right), \quad r \leq |\xi| \leq 1,\]

or more explicitly

\[(3.5)\quad f_2(\xi) = f_2(\omega(\xi)) = \frac{C(\Gamma + 1)}{2} \left[ q \ln \frac{\xi - \xi_1^{-1}}{\xi - \xi_2^{-1}} + \sum_{n=1}^{+\infty} (a_n \xi^n + a_{-n} \xi^{-n}) \right] + \frac{\bar{C}(\Gamma - 1)}{2} \left[ \bar{q} \ln \frac{\bar{\xi} - \bar{\xi}_1}{\bar{\xi} - \bar{\xi}_2} + \sum_{n=1}^{+\infty} (\bar{a}_n \xi^n + \bar{a}_{-n} \xi^{-n}) \right], \quad r \leq |\xi| \leq 1.\]

By enforcing the traction-free condition along the hole boundary $L_2$ in Eq. (3.1b),
the following relationships can be obtained

\begin{equation}
(3.6) \quad a_{-n} = \frac{\bar{C}[1 - \Gamma - r^{2n}(\Gamma + 1)]}{C[\Gamma + 1 + r^{2n}(\Gamma - 1)]} \bar{a}_n + \frac{r^{2n}[\frac{\bar{C}}{C}q(\Gamma + 1)(\xi_1^n - \xi_2^n) + q(\Gamma - 1)(\bar{\xi}_1^n - \bar{\xi}_2^n)]}{n[\Gamma + 1 + r^{2n}(\Gamma - 1)]}, \quad n = 1, 2, \ldots, +\infty,
\end{equation}

which demonstrates that \( a_{-n} \) is related to \( a_n \). It can be easily checked that in the absence of the screw dislocation dipole with \( q = 0 \), Eq. (3.6) simply reduces to the result by Milton and Serkov [8].

A comparison of Eq. (3.5) with Eq. (3.1c) leads to

\begin{equation}
(3.7) \quad q = \frac{b_3}{\pi C(\Gamma - 1)},
\end{equation}

which indicates that \( q \) is real if \( C \) is real (i.e., \( \sigma_{32}^\infty \neq 0, \sigma_{31}^\infty = 0 \)), and \( q \) is purely imaginary if \( C \) is purely imaginary (i.e., \( \sigma_{31}^\infty \neq 0, \sigma_{32}^\infty = 0 \)).

Examples of neutral coated holes are constructed in Figs. 3 and 4 with only \( a_1 \neq 0 \) in Eq. (3.6); in Figs. 5 and 6 with only \( a_1, a_2 \neq 0 \); in Figs. 7 and 8 with only \( a_1, a_3 \neq 0 \) in Eq. (3.6); in Figs. 9 and 10 with only \( a_1, a_4 \neq 0 \) in Eq. (3.6); in Fig. 11 with only \( a_1, a_5 \neq 0 \) in Eq. (3.6); and in Fig. 12 with only \( a_1, a_6 \neq 0 \).

![Figure 3](image-url)  

**Fig. 3.** Neutral coated hole when choosing \( \text{Re} C = 0, \Gamma = 1/3, r = 0.8, a_1 = 1, q = 0.25\text{i}, \xi_1 = 0.85, \xi_2 = 0.95 \).
Fig. 4. Neutral coated hole when choosing $\text{Im} C = 0$, $\Gamma = 1/3$, $r = 0.8$, $a_1 = 1$, $q = 0.5$, $\xi_1 = 0.85$, $\xi_2 = 0.95$.

Fig. 5. Neutral coated hole when choosing $\text{Re} C = 0$, $\Gamma = 1/3$, $r = 0.75$, $a_1 = 1 - i$, $a_2 = 0.3 - 0.3i$, $q = -i$, $\xi_1 = 0.8$, $\xi_2 = 0.95$. 
Fig. 6. Neutral coated hole when choosing $\text{Im}C = 0$, $\Gamma = 1/3$, $r = 0.75$, $a_1 = 1 - i$,
$a_2 = 0.3 - 0.3i$, $q = 0.5$, $\xi_1 = 0.8$, $\xi_2 = 0.95$.

Fig. 7. Neutral coated hole when choosing $\text{Re}C = 0$, $\Gamma = 1/3$, $r = 0.8$, $a_1 = 1$, $a_3 = 0.17$,
$q = 0.5i$, $\xi_1 = 0.85$, $\xi_2 = 0.95$. 
Fig. 8. Neutral coated hole when choosing $\text{Im}C = 0$, $\Gamma = 1/3$, $r = 0.8$, $a_1 = 1$, $a_3 = 0.17$, $q = 0.5$, $\xi_1 = 0.85$, $\xi_2 = 0.95$.

Fig. 9. Neutral coated hole when choosing $\text{Re}C = 0$, $\Gamma = 1/3$, $r = 0.75$, $a_1 = i$, $a_3 = -0.1i$, $q = 0.5i$, $\xi_1 = 0.8$, $\xi_2 = 0.95$. 
Fig. 10. Neutral coated hole when choosing $\text{Im} C = 0$, $\Gamma = 1/4$, $r = 0.75$, $a_1 = i$, $a_3 = -0.1i$, $q = -0.1$, $\xi_1 = 0.755i$, $\xi_2 = 0.995i$.

Fig. 11. Neutral coated hole when choosing $\text{Im} C = 0$, $\Gamma = 1/3$, $r = 0.8$, $a_1 = 1$, $a_5 = 0.1$, $q = 0.3$, $\xi_1 = 0.85$, $\xi_2 = 0.95$. 
in Eq. (3.6). The values of $a_n$ and $r$ in Figs. 5–10 are those adopted by Milton and Serkov [8] in the absence of the screw dislocation dipole. In all of these figures, the plus sign “+” indicates the position of the screw dislocation at $z = z_1$ whilst the star “∗” indicates that of the other screw dislocation with the opposite Burgers vector at $z = z_2$. It is observed from Figs. 3 and 4 that the neutral coated hole is no longer a confocal ellipse construction [26, 27] when a screw dislocation dipole is present in the coating although only $a_1$ is nonzero in Eq. (3.6). By comparing Fig. 5 with Fig. 1c in [8], Fig. 7 with Fig. 2b in [8], Fig. 9 with Fig. 3c in [8], one can conclude that the existence of the screw dislocation dipole in the thick coating will significantly alter the shapes of the neutral coated holes, especially those portions in the neighborhood of the screw dislocation dipole. As the index $n$ of the nonzero coefficient $a_n$ increases, the shape of the coated hole becomes more complicated (see Figs. 11 and 12).

We show in Fig. 13 neutral coated holes which are symmetric with respect to both the $x_1$- and $x_2$-axes. The positions of the two screw dislocations comprising the screw dislocation dipole are also symmetric with respect to the $x_2$-axis. It is seen that the outer boundary of the thick coating has been significantly influenced by the nearby screw dislocation dipole. In all the neutral coated holes shown in Figs. 3–13, the coating is always stiffer than the surrounding matrix (i.e., $Γ < 1$) in order to achieve neutrality.
4. Neutral coated holes in the presence of circular thermal inclusions

As shown in Fig. 14, we consider a domain in $\mathbb{R}^2$, infinite in extent, containing a hole occupying a simply connected region $\Omega$. The hole is surrounded by a thick coating occupying a doubly connected region $S_2$ with the coating assumed to be perfectly bonded to the surrounding matrix occupying the region $S_1$. The region occupying the coating-matrix interface is denoted by $L_1$ whilst the traction-free boundary of the hole is denoted by $L_2$. The matrix is subjected to uniform remote hydrostatic stresses $\sigma_{11}^\infty = \sigma_{22}^\infty = \sigma_0^\infty$, $\sigma_{12}^\infty = 0$. In addition, a circular region $|z - z_0| \leq R$ in the coating undergoes uniform stress-free eigenstrains $\varepsilon_{11}^* = \varepsilon_{22}^* = \varepsilon^*$, $\varepsilon_{12}^* = 0$. We again adopt the convention that the subscripts 1 and 2 identify the respective quantities in $S_1$ and $S_2$. Our objective is to design the shape of the thick coating such that the coated hole will not disturb the original uniform hydrostatic stress field in the surrounding matrix.
Fig. 14. A neutral coated hole with a circular thermal inclusion in the coating.

The boundary value problem for the two-phase composite takes the following form:

\[(4.1a)\] \( \phi_2(z) + z \phi'_2(z) + \psi_2(z) = \phi_1(z) + z \phi'_1(z) + \psi_1(z), \]
\( \kappa_2 \phi_2(z) - z \phi'_2(z) - \psi_2(z) = \Gamma \kappa_1 \phi_1(z) - \Gamma z \phi'_1(z) - \Gamma \psi_1(z), \quad z \in L_1; \)
\[(4.1b)\] \( \phi_2(z) + z \phi'_2(z) + \psi_2(z) = 0, \quad z \in L_2; \)
\[(4.1c)\] \( \phi_2(z) \cong O(1), \quad \psi_2(z) \cong -\frac{4 \mu_2 \varepsilon^*}{1 + \kappa_2} \frac{R^2}{z - z_0} + O(1), \quad z \to z_0; \)
\[(4.1d)\] \( \phi_1(z) = A z, \quad \psi_1(z) = 0, \quad z \in S_1, \)

where \( \Gamma = \mu_2/\mu_1 \), and the real constant \( A \) is related to remote loading through

\[(4.2)\] \[ A = \frac{\sigma_{11}^\infty + \sigma_{22}^\infty}{4} = \frac{\sigma_0^0}{2}. \]

In writing the asymptotic behavior in Eq. (4.1c), we have used the result given by Suo [28] and Ru [29]. A detailed explanation of Eq. (4.1c) is given in the Appendix. We emphasize that the definition of \( \Gamma \) used here for plane elasticity differs from that used in the case anti-plane elasticity discussed in the previous section.
Consider the following conformal mapping function for the thick coating

\[(4.3)\quad z = \omega(\xi) = \frac{q}{\xi - \xi_0^{-1}} + \sum_{n=1}^{+\infty} (a_n \xi^n + a_{-n} \xi^{-n}), \quad \xi = \omega^{-1}(z), \quad r \leq |\xi| \leq 1,\]

where \(q\) is a complex constant, \(a_n, a_{-n}, n = 1, 2, \ldots, +\infty\) are unknown complex constants to be determined and \(\xi_0 = \omega^{-1}(z_0)\). Using the mapping function in Eq. (4.3), the thick coating in the \(z\)-plane is mapped onto an annulus \(r \leq |\xi| \leq 1\) in the \(\xi\)-plane, the coating-matrix interface \(L_1\) in the \(z\)-plane is mapped onto \(|\xi| = 1\) in the \(\xi\)-plane, the traction-free boundary of the hole \(L_2\) in the \(z\)-plane is mapped onto \(|\xi| = r\) in the \(\xi\)-plane and the point \(z = z_0\) is mapped onto \(\xi = \xi_0\) (see Fig. 15). The appearance in Eq. (4.3) of the first-order pole at \(\xi = \xi_0^{-1}\) outside the annulus is to account for the existence of the circular thermal inclusion in the thick coating.

\[\text{Fig. 15. The problem in the image } \xi\text{-plane.}\]

By enforcing the continuity conditions of displacement and traction across the coating-matrix interface \(L_1\) (Eq. (4.1a)) we arrive at

\[(4.4)\quad \phi_2(\xi) = \phi_2(\omega(\xi)) = \frac{A[\Gamma(\kappa_1 - 1) + 2]}{\kappa_2 + 1} \left[ \frac{q}{\xi - \xi_0^{-1}} + \sum_{n=1}^{+\infty} (a_n \xi^n + a_{-n} \xi^{-n}) \right],\]

\[(4.4)\quad \psi_2(\xi) = \psi_2(\omega(\xi)) = \frac{2A[\kappa_2 - 1 - \Gamma(\kappa_1 - 1)]}{\kappa_2 + 1} \times \left[ -\frac{q\xi_0^2}{\xi - \xi_0} + \sum_{n=1}^{+\infty} (\bar{a}_n \xi^{-n} + \bar{a}_{-n} \xi^n) \right], \quad r \leq |\xi| \leq 1.\]
The traction-free boundary condition along the hole boundary $L_2$ (Eq. (4.1b)) allows us to determine the complex coefficients $a_n$, $a_{-n}$, $n = 1, 2, \ldots, +\infty$ uniquely as follows

\begin{align}
    a_n &= \frac{q\xi_0^2}{1 + \beta r^{-2n}}, \quad a_{-n} = -\frac{q\beta r^{2n}\xi_0^{-n}}{1 + \beta r^{2n}}, \quad n = 1, 2, \ldots, +\infty,
\end{align}

where

\begin{equation}
    \beta = \frac{\kappa_2 - 1 - \Gamma(\kappa_1 - 1)}{\Gamma(\kappa_1 - 1) + 2}.
\end{equation}

A comparison of Eq. (4.4) with Eq. (4.1c) leads to the following relationship

\begin{equation}
    q = \frac{4R^2\mu_2\varepsilon^*}{\sigma^0[\kappa_2 - 1 - \Gamma(\kappa_1 - 1)]} \frac{1}{\xi_0^2\omega'(\xi_0)}.
\end{equation}

Inserting Eq. (4.5) into Eq. (4.3), we obtain the following expression for $\omega'(\xi_0)$:

\begin{equation}
    \omega'(\xi_0) = -\frac{q\xi_0^2}{(|\xi_0|^2 - 1)^2} + q\xi_0^2 \sum_{n=1}^{+\infty} n \left( \frac{|\xi_0|^{2n-2}}{1 + \beta r^{-2n}} + \frac{\beta r^{2n}|\xi_0|^{-2n-2}}{1 + \beta r^{2n}} \right).
\end{equation}

Consequently, $\omega'(\xi_0)$ is uniquely determined from Eq. (4.8) for given values of $\Gamma, \kappa_1, \kappa_2, r, \xi_0$ and $q$. Accordingly, the ratio $R^2\mu_2\varepsilon^*/\sigma^0$ can be further determined from Eq. (4.7) as follows

\begin{equation}
    \frac{R^2\mu_2\varepsilon^*}{\sigma^0} = \frac{1}{4} \frac{|q|^2|\xi_0|^2[\kappa_2 - 1 - \Gamma(\kappa_1 - 1)]}{\xi_0^2\omega'(\xi_0)} \times \left[ -\frac{|\xi_0|^2}{(|\xi_0|^2 - 1)^2} + \sum_{n=1}^{+\infty} n \left( \frac{|\xi_0|^{2n}}{1 + \beta r^{-2n}} + \frac{\beta r^{2n}|\xi_0|^{-2n-2}}{1 + \beta r^{2n}} \right) \right].
\end{equation}

Note that the right-hand side of Eq. (4.9) is always real-valued.

In addition, we see from Eqs. (4.2)–(4.4) and the Appendix that the mean stress is uniform in the region $S'_2$ occupied by the thick coating outside the circular thermal inclusion and is given by

\begin{equation}
    \sigma_{11} + \sigma_{22} = \frac{2\sigma^0[\Gamma(\kappa_1 - 1) + 2]}{\kappa_2 + 1}, \quad z \in S'_2.
\end{equation}

It then follows from Eq. (4.10) that the hoop stress is constant along either $L_1$ or $L_2$ on the coating side as follows

\begin{equation}
    \sigma_{tt} = \frac{2\sigma^0[\Gamma(\kappa_1 - 1) + 2]}{\kappa_2 + 1}, \quad z \in L_1;
\end{equation}

\begin{equation}
    \sigma_{tt} = \frac{2\sigma^0[\Gamma(\kappa_1 - 1) + 2]}{\kappa_2 + 1}, \quad z \in L_2.
\end{equation}
Fig. 16. A neutral coated hole when choosing $\Gamma = 1.2582$, $\kappa_1 = \kappa_2 = 2$, $r = 0.3$, $\xi_0 = 0.65$, $q = 1$.

Fig. 17. A neutral coated hole when choosing $\Gamma = 2$, $\kappa_1 = \kappa_2 = 2$, $r = 0.3$, $\xi_0 = 0.65$, $q = 1$.

Thus the “equal strength” design criterion advanced by Cherepanov [30] has also been achieved.
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Illustrated in Figs. 16 to 21 are the resulting neutral coated holes for three typical values of the coating thickness parameter $r = 0.3, 0.6, 0.9$. The star in
Fig. 20. A neutral coated hole when choosing $\Gamma = 13.6492$, $\kappa_1 = \kappa_2 = 2$, $r = 0.9$, $\xi_0 = 0.95$, $q = 1$.

Fig. 21. A neutral coated hole when choosing $\Gamma = 16$, $\kappa_1 = \kappa_2 = 2$, $r = 0.9$, $\xi_0 = 0.95$, $q = 1$.

Each figure indicates the position of $z = z_0$, the center of the circular thermal inclusion. In all the six figures, the coating is always stiffer than the surrounding matrix (i.e., $\Gamma > 1$) in order to achieve neutrality. In addition, in Figs. 16 and 18,
the hole boundary $L_2$ is convex whereas the coating-matrix interface $L_1$ becomes non-convex and has a sharp corner; in Fig. 17, the coating-matrix interface $L_1$ is convex whereas the hole boundary $L_2$ becomes non-convex; in Figs. 19–21, both the coating-matrix interface $L_1$ and the hole boundary $L_2$ become non-convex. We observe the general trend that as the value of the parameter $r$ increases, the minimum value of $\Gamma (> 1)$ must increase in order to ensure that there is no self-intersecting boundary for $L_1$ and $L_2$ (see Figs. 16, 18, 20). On the other hand, the value of $\Gamma$ cannot be set arbitrarily large. The above trend can be more clearly observed in Fig. 22. The pair $(r, \Gamma)$ must lie between the two curves in Fig. 22. We further note that the results in Fig. 22 are valid for any value of $q$.

![Graph](image)

**Fig. 22.** The range of permissible $\Gamma$ for different values of the parameter $r$ with $\xi_0 = (1 + r)/2$, $\kappa_1 = \kappa_2 = 2$.

Furthermore, by using the parameters used in Figs. 16–21, from Eq. (4.9) we calculate that, respectively

\begin{equation}
R^2 \mu_2 \varepsilon^* = -0.06, \ 0.1695, \ -2.7231, \ 3.9898, \ -659.7846, \ 779.8345.
\end{equation}

We can see from the above that $\varepsilon^*$ and $\sigma^0$ may have either the same or opposite signs.
5. Conclusions

Using conformal mapping techniques, we demonstrate that neutral coated holes in anti-plane elasticity continue to be available even in the presence of a screw dislocation dipole in the thick coating. A logarithmic function is introduced into the mapping function (3.3) to account for the existence of the screw dislocation dipole. Numerical examples demonstrate the feasibility of the design method. The present method can be easily modified to study the more general case in which the thick coating is under the action of an arbitrary number of screw dislocations with the sum of the Burgers vectors of these dislocations being zero.

Also by means of conformal mapping, we have successfully designed neutral coated holes when a circular thermal inclusion is present in the thick coating and when the matrix is subjected to uniform hydrostatic stresses at infinity. All the unknown coefficients appearing in the mapping function (4.3) are determined quite simply by Eq. (4.5). Numerical results demonstrate the feasibility of our design method. Finally, our results support the conjecture that when a thermal inclusion of arbitrary shape is present in the thick coating, the coated hole can still be made neutral to an isotropic field.

Appendix

The continuity conditions of traction and displacement across the perfect circular interface \( L_3 : |z - z_0| = R \) can be expressed as follows

\[
\begin{align*}
\phi_2(z) + z\overline{\phi_2(z)} + \psi_2(z) &= \phi_3(z) + z\overline{\phi_3(z)} + \psi_3(z), \\
\kappa_2\phi_2(z) - z\overline{\phi_2(z)} - \psi_2(z) &= \kappa_2\phi_3(z) - z\overline{\phi_3(z)} - \psi_3(z) + 2\mu_2\varepsilon^*z, \quad z \in L_3,
\end{align*}
\]

where \( \phi_3(z) \) and \( \psi_3(z) \) are the two complex potentials defined in the circular thermal inclusion whilst \( \phi_2(z) \) and \( \psi_2(z) \) are the two complex potentials defined in the region occupied by the thick coating outside the circular thermal inclusion, here denoted by \( S'_2 \).

After some simple operations, Eq. (A.1) can be rewritten as

\[
\begin{align*}
\phi_2(z) &= \phi_3(z) + \frac{2\mu_2\varepsilon^*z}{\kappa_2 + 1}, \\
\psi_2(z) + \frac{4\mu_2\varepsilon^*}{\kappa_2 + 1} \frac{R^2}{z - z_0} &= \psi_3(z) - \frac{4\mu_2\varepsilon^*\overline{z}_0}{\kappa_2 + 1}, \quad z \in L_3.
\end{align*}
\]
By considering the above, we construct the following two auxiliary functions

\[
\Phi(z) = \begin{cases} 
\phi_2(z), & z \in S'_1, \\
\phi_3(z) + \frac{2\mu_2 \varepsilon^* z}{\kappa_2 + 1}, & |z-z_0| \leq R,
\end{cases}
\]

(A.3)

\[
\Psi(z) = \begin{cases} 
\psi_2(z) + \frac{4\mu_2 \varepsilon^*}{\kappa_2 + 1} \frac{R^2}{z-z_0}, & z \in S'_2, \\
\psi_3(z) - \frac{4\mu_2 \varepsilon^* \bar{z}_0}{\kappa_2 + 1}, & |z-z_0| \leq R.
\end{cases}
\]

It is seen from Eqs. (A.2) and (A.3) that \(\Phi(z)\) and \(\Psi(z)\) are continuous across \(L_3\) and then analytic in \(S_2\), the region occupied by the thick coating.

The definitions in Eq. (A.3) also suggest that the region of definition of the two analytic functions \(\phi_2(z)\) and \(\psi_2(z)\) can be extended by analytic continuation to the circular domain \(|z-z_0| \leq R\) through the following:

\[
\phi_2(z) = \Phi(z), \quad \psi_2(z) = \Psi(z) - \frac{4\mu_2 \varepsilon^*}{\kappa_2 + 1} \frac{R^2}{z-z_0}, \quad z \in S_2,
\]

(A.4)

which indicates that \(\phi_2(z)\) is analytic in \(S_2\) whereas \(\psi_2(z)\) is meromorphic in \(S_2\) (there is a first-order pole at \(z = z_0\)).

Equation (4.1c) then follows accordingly. Here we point out that this technique has been adopted by Suo [28]. We can also use the relationship in Eq. (A.3) to arrive at \(\phi_3(\xi) = \phi_3(\omega(\xi))\) and \(\psi_3(\xi) = \psi_3(\omega(\xi))\) once \(\phi_2(\xi)\) and \(\psi_2(\xi)\) have been determined.

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References


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