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Plane receding contact problem for a functionally graded layer supported by two quarter-planes

İ. ÇÖMEZ[1], V. KAHYA[1], R. ERDÖL[2]

[1] *Faculty of Engineering*
Department of Civil Engineering
Karadeniz Technical University
61080 Trabzon, Turkey
e-mail: isacomez@ktu.edu.tr

[2] *Faculty of Engineering and Architecture*
Department of Civil Engineering
Avrasya University
61010 Trabzon, Turkey

In this study, the plane receding contact problem for a functionally graded (FG) layer resting on two quarter-planes is considered by using the theory of linear elasticity. The layer is indented by a rigid cylindrical punch that applies a concentrated force in the normal direction. While the Poisson's ratio is kept constant, the shear modulus is assumed to vary exponentially through-the-thickness of the layer. It is assumed that the contact at the layer-punch interface and the layer-substrate interface is frictionless, and only the normal tractions can be transmitted along the contact regions. Applying the Fourier integral transform, the plane elasticity equations are converted to a system of two singular integral equations, in which the contact stresses and the contact widths are unknowns. The singular integral equations are solved numerically by Gauss–Jacobi integration formula. Effects of the material inhomogeneity, the distance between quarter-planes and the punch radius on the contact stresses, the contact widths, and the stress intensity factors at the sharp edges are shown. Although the theoretical analysis is formulated with respect to elastic quarter planes, the numerical studies are carried out only for rigid ones.

Key words: receding contact, functionally graded material, quarter-planes, stress intensity factor, singular integral equation.

Notation

a	contact width under the punch,
b, c	contact widths between the layer and quarter planes,
h	layer thickness,
$k(c)$	stress intensity factor,
N	collocation points,
$p_1(x)$	contact pressure under the punch,
$p_2(x)$	contact pressure between the layer and quarter plane,

P concentrated normal force,
R radius of the cylindrical punch,
W^N weighting constant,
γ inhomogeneity parameter,
μ shear modulus,
ν Poisson's ratio,
$\phi(x)$ dimensionless contact pressure,
Γ Gamma function,
χ shear modulus ratio of the layer (top surface to the bottom surface),
Ω, Ψ inverse Fourier transforms of the displacement.

1. Introduction

Since the most structural and mechanical system components are in contact with each other, contact mechanics finds a wide practical application in the field of solid mechanics. The stress distribution and deformation at the contact surfaces play a fundamental role on the behavior of engineering structures such as road pavements, railway ballasts, foundations, brake disks and abradable seals in gas turbine components.

When two bodies are in contact without bond upon loading, the contact zone shrinks as the bodies are deformed, and the initial contact area decreases to a finite size. This type of contact is called as receding contact [1]. There are numerous studies focusing on the receding contact problems related to the homogeneous solids in the literature. For example, the axisymmetric receding contact problem for an elastic plate pressed against the half-space with a concentrated force was studied by Weitsman [2] and Pu and Hussain [3]. Keer *et al.* [4] studied both plane and axisymmetric receding contact problems between a layer and a half space pressed with concentrated and uniformly distributed loads. Axisymmetric double receding contact problems were evaluated by Civelek and Erdogan [5] and Geçit [6]. Ratwani and Erdogan [7], Comez *et al.* [8], Kahya *et al.* [9] and Adibelli *et al.* [10] studied plane double receding contact problems. Above mentioned researchers neglected the frictional forces at the contact surfaces, and they assumed only that compressive normal tractions can be transmitted through the contact surface. On the other side, Çömez [11] investigated the frictional effect on the double receding contact problem.

Functionally graded materials (FGM) are inhomogeneous composites that the volume fractions of constituent materials vary gradually along spatial direction. Contact problems involving FG layer/substrate have been studied increasingly in recent years because such materials are widely used in load transfer components. Giannakopoulos and Pallot [12], Dag *et al.* [13], and Chen *et al.* [14] studied the frictional contact problems of a rigid punch on an FG half-plane. In these studies, the elastic moduli of the half-plane was assumed to vary with depth according to the power-law [12], or in exponential form either in a lateral direction [13], or in an arbitrary direction [14]. Guler and Erdogan [15], Liu *et al.* [16], and Chen and Chen [17] investigated the contact problem of a homogeneous half-space coated with exponential or linear graded layers. Guler and Erdogan [15] and Chen and Chen [17] considered the frictional plane contact problem by using the Fourier transform, while Liu *et al.* [16] studied on the frictionless axisymmetric contact problem by using the Hankel transform. Yang and Ke [18] and Chidlow *et al.* [19] examined the frictionless and frictional contact problems between a coating-graded layer-substrate and a rigid punch, respectively. The thermoelastic frictional contact and contact instability problems of FG layer were studied by Liu *et al.* [20] and Mao *et al.* [21, 22].

The receding contact problem of an FG layer resting on a half-plane was considered by El-Borgi and colleagues. They studied the frictionless plane problem when the components were pressed by normal tractions [23]. In another study, these researchers extended the frictionless problem to the frictional case [24]. Rhimi *et al.* [25] studied the axisymmetric contact problem when the system was pressed by a rigid punch. Çömez *et al.* [26] studied the double contact problem of FG bi-layer bonded to the rigid substrate. El-Borgi and Çömez [27] examined the frictional receding contact problem of a FG layer resting on a homogeneous half plane. Yan and Li [28] studied the double receding plane contact problem for an FG layer resting on an elastic homogeneous layer. In their studies, Yan and Mi [29, 30] examined the FG coated homogeneous layer or homogeneous coated FG layer resting on a homogeneous half plane.

If one of the contacting bodies has a sharp edge at the boundary of contact surface, locally high stresses can be anticipated [31]. Such a problem arises, for instance, in contact problems of quarter-planes. By using the reflection method, Hetenyi [32] found a solution for quarter space subjected to a normal point load. Based on the Hetenyi's reflection method, the first numerical investigation related to the frictionless flat punch problem on a quarter-plane and a quarter space was carried out by Gerber [33]. The plane problem of two dissimilar cylindrical composite wedges was discussed by Boggy and Wang [34] using the Mellin transforms, and they developed the stress singularities at the corner for the material constants and corner angle. In the framework of the Fourier and Mellin transforms, Erdogan and Ratwani [35] examined the receding contact problem of an elastic layer supported by two elastic quarter-planes. Bakioglu [36] studied the contact problem of an elastic layer pressed to the elastic wedge with a rigid cylindrical punch. By using the integral transform technique, Keer *et al.* [37] solved the frictionless contact problem between a quarter space and a rigid cylinder. The non-symmetrical contact problem of an elastic layer supported with two elastic quarter-planes was examined both analytically and numerically by Aksogan *et al.* [38]. Yaylaci and Birinci [39] considered the contact problem of two homogeneous layers supported by two elastic quarter-planes.

The receding contact problem of FG layer resting on two quarter planes was examined by Adiyaman *et al.* [40]. The top of the layer is subjected to a distributed load. In the present study the prescribed traction load is replaced by a cylindrical rigid punch. Since FGM coatings have a potential application to improve the properties of surfaces in contact, the layer is modeled as FG instead of homogeneous. The FG layer is pressed to the quarter-planes through a rigid cylindrical punch that applies a concentrated force in the normal direction. The shear modulus is assumed to vary exponentially through-the-thickness of the layer. By using the Fourier transform, the plane elasticity equations are converted to a system having two singular integral equations, in which the unknowns are the contact stresses and the contact widths. The singular integral equations are then solved numerically by the Gauss–Jacobi integration formulae. The main objective of the study is to examine the contact stresses, the contact widths, and the stress intensity factors at sharp edges depending on the material inhomogeneity, the distance between the quarter-planes and the punch radius. Note that the problem is formulated with respect to elastic quarter planes however the numerical studies are performed only for rigid ones.

2. Formulation of the problem

The geometry of the two-dimensional (2D) frictionless contact problem is shown in Fig. 1. FG elastic layer of thickness (h)is supported by two quarter-planes. The rigid cylindrical punch with radius (R) is pressed against the layer by a concentrated normal force (P). a and b refers to the contact half-widths between the rigid punch and the FG layer, and between the FG layer and quarter-

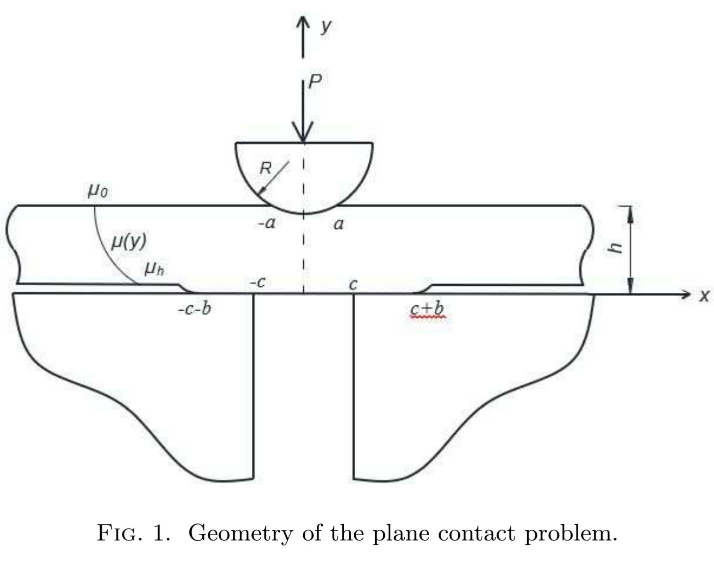

Fig. 1. Geometry of the plane contact problem.

planes, respectively. The distance between the quarter-planes is defined as $2c$. The Poisson's ratio (ν) is taken to be a constant while the shear modulus of the layer (μ) varies exponentially according to the following law:

$$\mu_1(y) = \mu_0 e^{\gamma y}, \qquad \gamma h = \ln\left(\frac{\mu_h}{\mu_0}\right). \tag{2.1}$$

where γ is the material gradation parameter, μ_0 and μ_h are the shear moduli on the bottom and top surfaces of the layer, respectively.

The stress-displacement relations of the FG layer can be written as:

$$\sigma_{x1}(x,y) = \frac{\mu_1(y)}{\kappa_1 - 1}\left[(\kappa_1+1)\frac{\partial u_1(x,y)}{\partial x} + (3-\kappa_1)\frac{\partial v_1(x,y)}{\partial y}\right], \tag{2.2}$$

$$\sigma_{y1}(x,y) = \frac{\mu_1(y)}{\kappa_1 - 1}\left[(3-\kappa_1)\frac{\partial u_1(x,y)}{\partial x} + (\kappa_1+1)\frac{\partial v_1(x,y)}{\partial y}\right], \tag{2.3}$$

$$\tau_{xy1} = \mu_1(y)\left[\frac{\partial u_1(x,y)}{\partial y} + \frac{\partial v_1(x,y)}{\partial x}\right], \tag{2.4}$$

where $u_1(x,y)$, $v_1(x,y)$ are the x- and y- components of the displacements and $\kappa_1 = 3 - 4\nu_1$ for the plane strain model.

The governing partial differential equations for the plane contact problem can be given as:

$$(\kappa_1+1)\frac{\partial^2 u_1}{\partial x^2} + (\kappa_1-1)\frac{\partial^2 u_1}{\partial y^2} + 2\frac{\partial^2 v_1}{\partial x \partial y} + \gamma(\kappa_1-1)\left[\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right] = 0, \tag{2.5}$$

$$(\kappa_1-1)\frac{\partial^2 v_1}{\partial x^2} + (\kappa_1+1)\frac{\partial^2 v_1}{\partial y^2} + 2\frac{\partial^2 u_1}{\partial x \partial y} + \gamma\left[(3-\kappa_1)\frac{\partial u_1}{\partial x} + (\kappa_1+1)\frac{\partial v_1}{\partial y}\right] = 0. \tag{2.6}$$

Equations (2.5) and (2.6) can be solved by using the Fourier transforms which are defined as follows:

$$u_1(x,y) = \frac{2}{\pi}\int_0^\infty \Omega(\alpha,y)\sin(\alpha x)\,d\alpha, \tag{2.7}$$

$$v_1(x,y) = \frac{2}{\pi}\int_0^\infty \psi(\alpha,y)\cos(\alpha x)\,d\alpha, \tag{2.8}$$

where α is the transform variable, $\Omega(\alpha,y)$ and $\psi(\alpha,y)$ are the Fourier transforms of $u_1(x,y)$ and $v_1(x,y)$, respectively. By substituting Eqs. (2.7) and (2.8) into Eqs. (2.5) and (2.6), the following partial differential equations can be obtained:

$$-(\kappa_1+1)\alpha^2\Omega + (\kappa_1-1)\frac{d^2\Omega}{dy^2} - 2\alpha\frac{d\psi}{dy} + \gamma(\kappa_1-1)\left[\frac{d\Omega}{dy} - \alpha\psi\right] = 0, \tag{2.9}$$

$$-(\kappa_1-1)\alpha^2\psi + (\kappa_1+1)\frac{d^2\psi}{dy^2} + 2\alpha\frac{d\Omega}{dy} + \gamma\left[(3-\kappa_1)\alpha\Omega + (\kappa_1+1)\frac{d\psi}{dy}\right] = 0. \tag{2.10}$$

Through applying some mathematical manipulations, Eqs. (2.9) and (2.10) can be combined in the following ordinary differential equation for Ω:

$$\frac{d\Omega^4}{dy^4}+2\gamma\frac{d\Omega^3}{dy^3}+(\gamma^2-2\alpha^2)\frac{d\Omega^2}{dy^2}-2\alpha^2\gamma\frac{d\Omega}{dy}+\alpha^2\left(\alpha^2+\gamma^2\frac{3-\kappa_1}{1+\kappa_1}\right)\Omega=0, \tag{2.11}$$

solution of which can be written as

$$\Omega(\alpha,y)=\sum_{j=1}^{4}A_j e^{n_j y}, \tag{2.12}$$

where n_j is defined by two complex conjugates as $n_3=\bar{n}_1$, $n_4=\bar{n}_2$, and A_j refers to the unknown coefficients to be determined from the boundary conditions of the problem; n_1 and n_2 are given as

$$n_1=-\frac{1}{2}\left(\gamma+\sqrt{4\alpha^2+\gamma^2+4I\alpha|\gamma|\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right), \tag{2.13}$$

$$n_2=-\frac{1}{2}\left(\gamma-\sqrt{4\alpha^2+\gamma^2+4I\alpha|\gamma|\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right). \tag{2.14}$$

By substituting Eq. (2.12) into Eqs. (2.9) and (2.10), $\psi(\alpha,y)$ can be obtained as

$$\psi(\alpha,y)=\sum_{j=1}^{4}A_j m_j e^{n_j y}, \tag{2.15}$$

where

$$m_j=\frac{(2n_j+\gamma(3-\kappa_1))(n_j^2+\gamma n_j-\alpha^2\left(\frac{3+\kappa_1}{1+\kappa_1}\right))}{\alpha(4\alpha^2+\gamma^2(3-\kappa_1))}. \tag{2.16}$$

Through inserting Eqs. (2.12) and (2.15) into Eqs. (2.17) and (2.18), the displacement components for the FG layer can readily be obtained as follows:

$$u_1(x,y)=\frac{2}{\pi}\int_0^\infty\sum_{j=1}^{4}A_j e^{n_j y}\sin(\alpha x)d\alpha, \tag{2.17}$$

$$v_1(x,y)=\frac{2}{\pi}\int_0^\infty\sum_{j=1}^{4}A_j m_j e^{n_j y}\cos(\alpha x)d\alpha. \tag{2.18}$$

Substituting Eqs. (2.17) and (2.18) into Eqs. (2.2)–(2.4) yields the expressions of the stress field in the FG layer as follows:

$$\sigma_{x1}(x,y) = \frac{2}{\pi}\int\limits_0^\infty \sum_{j=1}^4 \frac{\mu_1(y)}{\kappa_1-1}[(3-\kappa_1)m_j n_j+\alpha(\kappa_1+1)]A_j e^{n_j y}\cos(\alpha x)d\alpha, \tag{2.19}$$

$$\sigma_{y1}(x,y) = \frac{2}{\pi}\int\limits_0^\infty \sum_{j=1}^4 \frac{\mu_1(y)}{\kappa_1-1}[(\kappa_1+1)m_j n_j+\alpha(3-\kappa_1)]A_j e^{n_j y}\cos(\alpha x)d\alpha, \tag{2.20}$$

$$\tau_{xy1}(x,y) = \frac{2}{\pi}\int\limits_0^\infty \sum_{j=1}^4 \mu_1(y)[(n_j-\alpha m_j)]A_j e^{n_j y}\sin(\alpha x)d\alpha. \tag{2.21}$$

3. Boundary conditions and singular integral equations

Due to symmetry, only half of the problem can be considered. Therefore, we can write the boundary conditions of the considered plane contact problem as follows:

$$\sigma_{y1}(x_1,h) = \begin{cases} -p_1(x_1) & 0 < x_1 < a, \\ 0 & a \le x_1 < \infty, \end{cases} \tag{3.1}$$

$$\tau_{xy1}(x_1,h) = 0, \quad 0 \le x_1 < \infty, \tag{3.2}$$

$$\sigma_{y1}(x_2,0) = \begin{cases} -p_2(x_2) & c < x_2 < c+b, \\ 0 & c+b \le x_2 < \infty, \end{cases} \tag{3.3}$$

$$\tau_{xy1}(x_2,0), = 0 \quad 0 \le x_2 < \infty, \tag{3.4}$$

where $p_1(x_1)$ and $p_2(x_2)$ are the unknown contact stresses under the punch and between the FG layer and the quarter-planes, respectively.

Applying the Fourier transforms to the boundary conditions given by Equation (3.1–3.4) with considering Equations (2.17), (2.18), and (2.19)–(2.21) provides a linear algebraic system of equations. Solving these equations gives the unknown functions A_j in terms of the Fourier transforms of the unknown tractions $p_1(x_1)$ and $p_2(x_2)$ as follows:

$$\begin{aligned} A_j = \frac{1}{\mu_0}A_{j1}\int\limits_0^a p_1(x_1)\cos(\alpha t_1)dt_1 \\ + A_{j2}\int\limits_c^{c+b} p_2(x_2)\cos(\alpha t_2)dt_2 \quad (j=1,\ldots,4), \end{aligned} \tag{3.5}$$

where A_{j1} and A_{j2} are the coefficients of $p_1(x_1)$ and $p_2(x_2)$ (the expressions of $p_1(x_1)$ and $p_2(x_2)$ are not given here since they are very long). The unknown contact pressures $p_1(x_1)$ and $p_2(x_2)$ can be determined from the following mixed boundary conditions related to the vertical displacements on the contact areas:

$$\frac{\partial v_1(x_1,h)}{\partial x_1} = \frac{x_1}{R} \quad (0 < x_1 < a), \tag{3.6}$$

$$\frac{\partial}{\partial x_2}[v_1(x_2,0) - v_2(x_2,0)] = 0 \quad (c < x_2 < c+b), \tag{3.7}$$

where v_2 is the vertical displacement of the quarter-plane. The above conditions, written in the derivative form, ensure the continuity of the normal displacement but eliminate the rigid-body displacements.

Using the Airy stress function and the Mellin transform, the vertical displacement on the top surface of quarter-plane can be written as follows [35]:

$$\frac{\partial v_2(x_2,0)}{\partial x_2} = -\frac{\kappa_2+1}{4\mu_2}\int\limits_c^{c+b} p_2(t_2)\bar{k}_{23}(x_2,t_2)dt_2, \tag{3.8}$$

where

$$\begin{aligned}\bar{k}_{23}(x_2,t_2) &= \\ &\frac{1}{x_2-c}\Bigg\{\int\limits_0^\infty\left(\frac{\sinh(\pi y)}{-2y^2-1+\cosh(\pi y)}-1\right)\sin\left[\log\left(\frac{x_2-c}{t_2-c}\right)y\right]dy \\ &\qquad + \frac{1}{\log\left(\frac{x_2-c}{t_2-c}\right)} - \frac{\pi^2}{\pi^2-4}\Bigg\}.\end{aligned} \tag{3.9}$$

Substituting the unknown functions A_j into the mixed conditions in (3.6) and (3.7), and extracting the Cauchy singularities from the kernels give the following system of singular integral equations:

$$\frac{1}{\pi}\int\limits_{-a}^{a} p_1(t_1)\left[\frac{1}{t_1-x_1}+k_{11}(x_1,t_1)\right]dt_1 + \frac{1}{\pi}\int\limits_c^{c+b} p_2(t_2)k_{12}(x_1,t_2)dt_2 = \frac{\mu_{01}}{\varphi_1}\frac{x_1}{R}, \tag{3.10}$$

$$\frac{1}{\pi}\int\limits_{-a}^{a} p_1(t_1)k_{21}(x_2,t_1)dt_1 + \frac{1}{\pi}\int\limits_c^{c+b} p_2(t_2)\left[\frac{1}{t_2-x_2}+k_{22}(x_2,t_2)+k_{23}(x_2,t_2)\right]dt_2 = 0. \tag{3.11}$$

The expressions for $k_{11}(x_1,t_1)$, $k_{12}(x_1,t_2)$, $k_{21}(x_2,t_1)$, $k_{22}(x_2,t_2)$, $k_{23}(x_2,t_2)$ and φ_1 are given in Appendix A. In addition to the contact pressures $p_1(x)$ and $p_2(x)$, the contact widths a and b are also unknown in the singular integral equations. To be able to complete the solution of the problem, the following global equilibrium conditions must be ensured by $p_1(x)$ and $p_2(x)$:

$$\int_{-a}^{a} p_1(t_1)dt_1 = P, \qquad \int_{c}^{c+b} p_2(t_2)dt_2 = \frac{P}{2}. \tag{3.12}$$

4. Numerical solution of the singular integral equations

By introducing the following dimensionless quantities:

$$s_1 = x_1/a, \qquad r_1 = t_1/a, \tag{4.1}$$

$$s_2 = 2(x_2 - c - b/2)/b, \qquad r_2 = 2(t_2 - c - b/2)/b, \tag{4.2}$$

$$\phi_1(r_1) = \frac{p_1(r_1)}{P/h}, \qquad \phi_2(r_2) = \frac{p_2(r_2)}{P/h}, \tag{4.3}$$

the singular integral Eqs. (3.10) and (3.11), and the equilibrium conditions (3.12) can be rewritten as

$$\frac{1}{\pi}\int_{-1}^{1} \phi_1(r_1)\left[\frac{1}{r_1 - s_1} + \frac{a}{h}k_{11}(s_1,r_1)\right]dr_1 + \frac{1}{\pi}\int_{-1}^{1} \phi_2(r_2)\,\frac{b}{2h}k_{12}(s_1,r_2)dr_2 = \frac{1}{\varphi_1}\,\frac{\mu_0/(P/h)}{R/h}\,\frac{a}{h}s_1, \tag{4.4}$$

$$\frac{1}{\pi}\int_{-1}^{1} \phi_1(r_1)\frac{a}{h}k_{21}(s_2,r_1)dr_1 + \frac{1}{\pi}\int_{-1}^{1} \phi_2(r_2)\left[\frac{1}{r_2 - s_2} + \frac{b}{2h}[k_{22}(s_2,r_2) + k_{23}(s_2,r_2)]\right]dr_2 = 0, \tag{4.5}$$

$$\frac{a}{h}\int_{-1}^{1} \phi_1(r_1)dr_1 = 1, \tag{4.6}$$

$$\frac{b}{h}\int_{-1}^{1} \phi_2(r_2)dr_2 = 1. \tag{4.7}$$

Due to the smooth contact at the end of the rigid cylindrical punch, the singular integral equation in Eq. (4.4) has an index -1; however, thanks to the geometry, the index of the integral equation in (4.5) is 0. The numerical method proposed by Erdogan [41] and Krenk [42] may be used to solve the system of integral equations given by Eqs. (4.4) and (4.5). In this method, the unknown functions $\phi_i(r_i)$ may be expressed as

$$\phi_1(r_1) = g_1(r_1)\sqrt{1-r_1^2}, \tag{4.8}$$

$$\phi_2(r_2) = g_2(r_2)(1-r_2)^{\alpha_2}(1+r_2)^{\beta_2}, \tag{4.9}$$

where $g_i(r_i)$ are the continuous and bonded functions in the interval $[-1,1]$. By using the Gauss–Jacobi quadrature formulae, the integral equations can be converted to the equivalent system of algebraic equations as follows:

$$\begin{aligned} &\sum_{i=1}^{N} W_{1i}^{N}\left[\frac{1}{r_{1i}-s_{1k}}+\frac{a}{h}k_{11}(s_{1k},r_{1m})\right]g_1(r_{1i}) \\ &+\sum_{i=1}^{N} W_{2i}^{N}\frac{b}{2h}k_{12}(s_{1k},r_{2i})g_2(r_{2i}) = \frac{1}{\beta_1}\frac{\mu_0/(P/h)}{R/h}\frac{a}{h}s_{1k}, \quad k=1,\dots,N+1, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\sum_{i=1}^{N} W_{1i}^{N}\frac{a}{h}k_{21}(s_{2k},r_{1i})g_1(r_{1i}) \\ &+\sum_{i=1}^{N} W_{2i}^{N}\left[\frac{1}{r_{2i}-s_{2k}}+\frac{b}{2h}[k_{22}(s_{2k},r_{2i})+k_{23}(s_{2k},r_{2i})]\right]g_2(r_{2i}) = 0, \quad k=1,\dots,N. \end{aligned} \tag{4.11}$$

Also, the equilibrium conditions in Eq. (3.12) become

$$\frac{a}{h}\sum_{i=1}^{N} W_{1i}^{N}\, g_1(r_{1i}) = \frac{1}{\pi}, \qquad \frac{b}{h}\sum_{i=1}^{N} W_{2i}^{N}\, g_2(r_{2i}) = \frac{1}{\pi}, \tag{4.12}$$

where r_{1i} and s_{1k} are the roots of the related Chebyshev polynomials, r_{2i} and s_{2k} are the roots of the related Jacobi polynomials, and W_{1i}^N and W_{2i}^N are the weighting constant as:

$$r_{1i} = \cos\left(\frac{i\pi}{N+1}\right) \quad i=1,\dots,N, \tag{4.13}$$

$$s_{1k} = \cos\left(\frac{\pi}{2}\frac{2k-1}{N+1}\right) \quad k=1,\dots,N+1, \tag{4.14}$$

$$P_N^{(\alpha_2,\beta_2)}(r_{2i}) = 0, \quad i=1,\dots,N, \tag{4.15}$$

$$P_N^{(-\alpha_2,-\beta_2)}(s_{2k}) = 0, \quad k=1,\dots,N, \tag{4.16}$$

$$W_{1i}^N = \frac{1 - r_{1i}^2}{N+1}, \tag{4.17}$$

$$W_{2i}^N = -\frac{1}{\pi}\frac{(2N+2+\alpha_2+\beta_2)\Gamma(N+1+\alpha_2)\Gamma(N+1+\beta_2)}{(N+1)!\Gamma(N+1+\alpha_2+\beta_2)} \times \frac{2^{\alpha_2+\beta_2}}{P_{N+1}^{(\alpha_2+\beta_2)}(r_{2i})\frac{d}{dr}P_N^{(\alpha_2,\beta_2)}(r_{2i})}, \tag{4.18}$$

where Γ refers to the Gamma function. It should be noted that there are $2N+1$ possible collocation points to determine the $2N$ unknowns $g_1(r_{1i})$ and $g_2(r_{2i})$. It can be shown that the $(N/2+1)^{th}$ equation in Eq. (4.10) is automatically satisfied. Thus, Eqs. from (4.10) to (4.12) give $2N+2$ equations in order to determine the $2N+2$ unknowns, which are $g_1(r_{1i})$, $g_2(r_{2i})$, a, and b. The system of equations is linear in terms of $g(r_i)$ but they are highly nonlinear in variables a and b. Therefore, an iterative method is used to identify unknowns. Firstly, initial values for a and b are estimated, then, the system of Eqs. (4.10) and (4.11) is solved for $g(r_i)$. Equations given in (4.12) are, then, verified in terms of the fact that the equilibrium conditions are satisfied or not. The iteration continues until the desired accuracy is obtained in the equations presented in (4.12).

5. Stress intensity factor

The stress intensity factor (SIF), which describes the stress state at a crack tip, is related to the rate of crack growth, and it is used to determine the failure criteria emerging due to the fracture. In terms of our problem, at the corner of the quarter-planes, the stresses in the FG layer have integrable singularity with the power α_2. The strength of the stress singularity or SIF gives some ideas about the intensity of stresses in the vicinity of the singular point [43].

After calculating $g_2(r_{2i})$, the SIF at $x_2 = c$ can be obtained through

$$k(c) = \lim_{x_2 \to c} \frac{p_2(x_2)b^{\beta_2}}{(x_2-c)^{\beta_2}2^{\alpha_2+\beta_2}} \tag{5.1}$$

or in dimensionless form

$$\frac{k(c)}{P/h} = g_2(-1). \tag{5.2}$$

For the latter equation, $g_2(-1)$ can be calculated as [44]

$$g_2(-1) = \sum_{m=0}^{N-1} d_m P_m^{(\alpha_2,\beta_2)}(-1), \tag{5.3}$$

where

$$(5.4)\qquad d_m = \frac{1}{h_m}\pi\sum_{i=1}^{N} W_{2i}^{N} P_m^{(\alpha_2,\beta_2)}(r_{2i})g_2(r_{2i}),$$

$$(5.5)\qquad h_m = 2^{\alpha_2+\beta_2+1}\frac{2N+2+\alpha_2+\beta_2}{(2m+\alpha_2+\beta_2+1)m!}\,\frac{\Gamma(m+\alpha_2+1)\Gamma(m+\beta_2+1)}{\Gamma(m+\alpha_2+\beta_2+1)}.$$

6. Results and discussion

In this study, some numerical results related to the contact widths, the contact stresses, and the SIFs depending on the punch radius, the distance between quarter-planes, and the material inhomogeneity are presented. Based on the study carried out by Erdogan and Gupta [43], $\alpha_2 = 0.5$ is assumed in Eqs. (3.1) and (3.4). In the contact problems, geometric singularities occur in the case that one of the contacting bodies has a sharp edge. The nature of the geometric singularities and the way to find β_2 for homogeneous elastic materials have been explained in frictionless cases by Dundurs and Lee [31] and Boggy and Wang [34] and in a frictional case by Comninou [45]. In the existing literature, the index values for homogeneous materials have been well-established. However, there is no study related to the geometric singularities and the indices for FGMs in the literature. Although the formulations given in previous sections are rather general, the numerical values are, thus, calculated only in case of rigid quarter-planes for simplicity, i.e., $\beta_2 = -0.5$.

Table 1 shows the convergency of the number of discrete points for numerical analyses carried out to calculate the contact areas. As it is seen, when the number of discrete points increases, a/h decreases while b/h increases. However, they almost remain constant after $N = 20$. The number of discrete points is taken as $N = 80$ for the required accuracy. In order to solve Eq. (2.4), the numerical integration of $k_{11}(x_1,t_1)$, $k_{12}(x_1,t_2)$, $k_{21}(x_2,t_1)$, $k_{22}(x_2,t_2)$, $k_{23}(x_2,t_2)$ are needed.

Table 1. The effect of the discrete point number on the contact areas ($c/h = 0.2$, $\mu_0/(P/h) = 100$, $\nu = 0.25$, $\mu_h/\mu_0 = 2$)a/h**.**

	$R/h = 100$		$R/h = 250$		$R/h = 500$	
Discrete points (N)	a/h	b/h	a/h	b/h	a/h	b/h
2	0.55270	0.77134	0.87675	0.92261	1.24011	1.12990
4	0.54952	0.78169	0.84752	0.93150	1.13078	1.13603
10	0.54775	0.78155	0.84628	0.93847	1.12918	1.14797
20	0.54722	0.78217	0.84752	0.94171	1.14203	1.15288
40	0.54722	0.78217	0.84747	0.94087	1.14058	1.15210
80	0.54722	0.78217	0.84745	0.94085	1.14014	1.15105

Table 2 shows the effect of the upper limit of integration on the contact areas. The contact areas almost remain constant after the upper limit 5. The upper limit is chosen as 50 for numerical integration, and Gauss–Kronrod quadrature is used.

Table 2. The effect of the upper limit of the integrals on the contact areas ($c/h = 0.2$, $R/h = 100$, $\mu_0/(P/h) = 100$, $\nu = 0.25$).

Integration upper limit	$\mu_h/\mu_0 = 0.5$		$\mu_h/\mu_0 \cong 1$		$\mu_h/\mu_0 = 2$	
	a/h	b/h	a/h	b/h	a/h	b/h
1	0.73951882	1.0494238	0.5773458	1.0566973	0.4428517	1.0988307
2	0.83617745	0.9278073	0.6651271	0.8667501	0.5110582	0.8555855
5	0.79770622	0.9236048	0.6645431	0.8253862	0.5438148	0.7839617
10	0.79567329	0.9172478	0.6644924	0.8267107	0.5450270	0.7831598
20	0.79507778	0.9207643	0.6644952	0.8267134	0.5457131	0.7784540
50	0.79675720	0.9201936	0.6657338	0.8273819	0.5473066	0.7778107
100	0.81612336	0.9377490	0.6828821	0.8367600	0.5594445	0.7854685

Figure 2 shows the variation of the normalized contact widths with μ_h/μ_0 for various c/h. Increasing of μ_h/μ_0, (i.e., the increase in the rigidity of the upper part), causes the decrease in both contact widths because the layer is more resistant to bending in this case. As seen in the figures, while c/h increases (i.e., the distance between quarter-planes increases), the contact width under the punch also increases since the layer is more affected by bending. On the contrary, the contact width at the interface of layer-quarter-plane decreases because the contact area at this region will inevitably be smaller when the layer more bends.

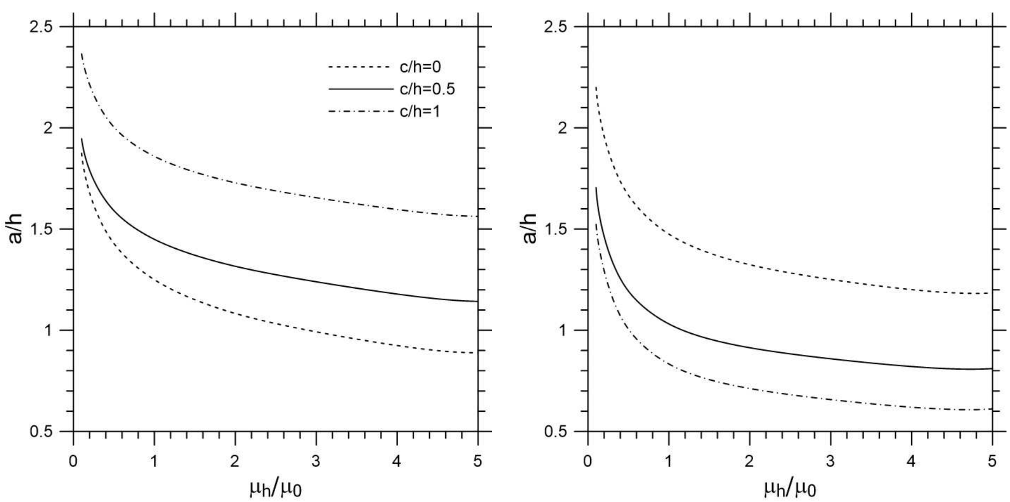

Fig. 2. Variation of the contact widths with μ_h/μ_0 for various c/h ($R/h = 500$, $\mu_0/(P/h) = 100$).

In Fig. 3, the contact pressures in a non-dimensional form are presented by depending onμ_h/μ_0. In this figure, it is seen that the non-dimensional contact stresses increase when μ_h/μ_0 increases since they distribute over a smaller contact area as explained in the preceding paragraph.

Figure 4 present the normalized contact stresses for various c/h. Since the contact widths under the punch increase with increasing the distance between quarter-planes, the contact stresses become smaller as seen. However, the contact

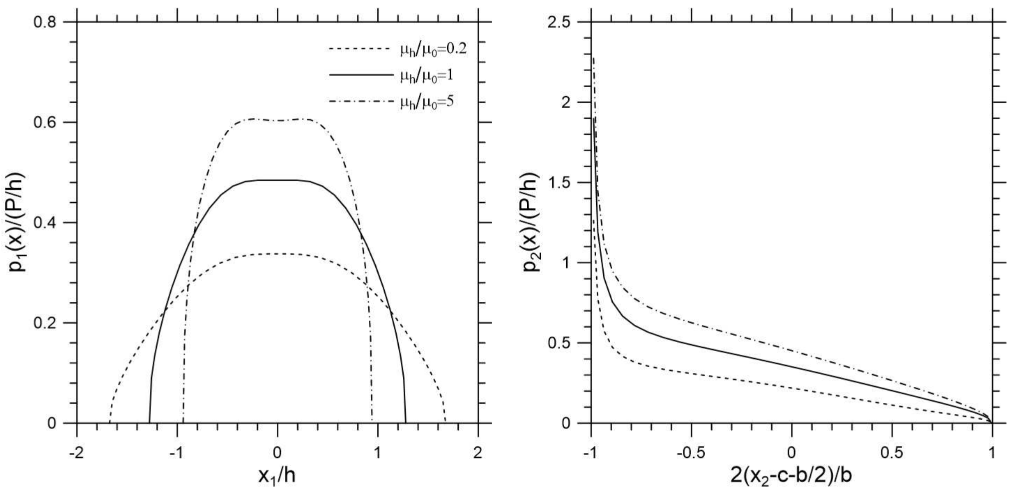

FIG. 3. Contact stress distributions under the punch and between the layer and the rigid quarter-planes for various values of μ_h/μ_0 ($c/h = 0.2$, $R/h = 500$, $\mu_0/(P/h) = 100$).

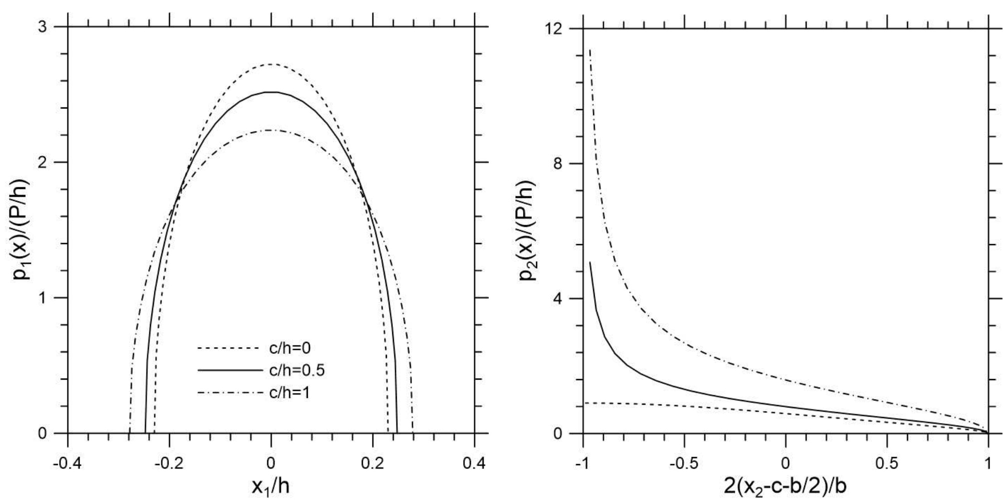

FIG. 4. Contact stress distributions under the punch and between the layer and the rigid quarter-planes for various c/h ($\mu_h/\mu_0 = 2$, $R/h = 500$, $\mu_0/(P/h) = 100$).

stresses at the layer-quarter-plane interface increase with increasing c/h since the contact area becomes smaller. When $c/h = 0$, the space between the quarter planes is closed, and a half plane is obtained. Thus, when $c/h = 0$, there is no singularity while the$x$ coordinate is approaching to -1.

Figure 5 shows the normalized contact stresses for various punch radii R/h. As the punch radius increases, the contact width under the punch also increases (see also Table 1); thus, the contact stresses become smaller since they distribute over a greater area. As it is seen, since the increase in R/h has less effect on b/h compared to a/h, the stresses at the interface between the layer-quarter-planes are not changed in a significant amount.

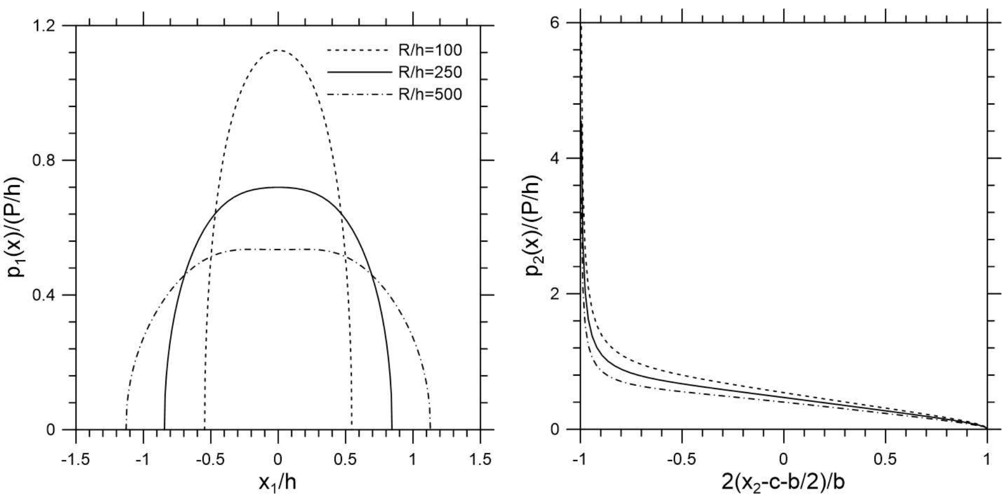

Fig. 5. Contact stress distributions under the punch and between the layer and the rigid quarter-planes for various R/h ($\mu_h/\mu_0 = 2$, $c/h = 0.2$, $\mu_0/(P/h) = 100$).

As can be seen in Figs. 3–5, in order for receding the contact stresses, they show a sharp increase in vicinity of the corners of quarter-planes. Table 3 presents the stress intensity factors (SIFs) at these corners for various μ_h/μ_0 and c/h. As can be observed from the table, SIFs increase when the top of the layer

Table 3. Stress intensity factors at the corners of rigid quarter-planes ($R/h = 500$, $\mu_0/(P/h) = 100$).

μ_h/μ_0	$k(c)/(P/h)$			
	$c/h = 0$	$c/h = 0.2$	$c/h = 0.5$	$c/h = 1$
0.2	0.00266	0.08860	0.16799	0.26274
0.5	0.00352	0.11754	0.20630	0.31258
1.0	0.00429	0.14325	0.24696	0.34724
2.0	0.00461	0.15288	0.24595	0.36978
5.0	0.00516	0.16656	0.25783	0.39205

becomes stiffer compared to its bottom; that is, μ_h/μ_0 increases. From the table, it can be also seen that as the distance between quarter-planes increases, SIF also increases. As explained in the preceding, when c/h increases, the receding contact area decreases since the layer more bends. This results in increasing of the contact stresses at this region, thus, SIFs also increases at the corners.

7. Conclusions

In this study, the receding contact problem of an FG layer resting on two quarter-planes by the classical theory of elasticity was presented. Using the Fourier integral transform, the plane elasticity equations were converted to a system of two singular integral equations, in which the contact stresses and contact widths were unknowns. The singular integral equations were numerically solved by the Gauss-Jacobi integration formulae. The effect of material inhomogeneity, the distance between the quarter-planes, and the punch radius on the contact stresses, the contact widths and the stress intensity factors at the sharp edges were studied. As a result of the study, we reached the following conclusions:

1. The material inhomogeneity described by μ_h/μ_0 has a significant effect on the contact widths and thus the contact stresses. When μ_h/μ_0 increases, the contact widths under the punch and at the layer-quarter-plane interface decrease. This results in that the contact stresses over these areas becomes greater.

2. The distance between quarter-planes is another important parameter on the contact widths and thus the contact stresses. When c/h increases, the contact area under the punch increases while the receding contact area decreases. This is exactly due to the increase in the layer's bending. This causes smaller contact stresses over the contact area under the punch, while greater stresses over the receding contact area.

3. The punch radius has great effects on the contact widths and thus corresponding contact stresses. However, R/h has little effect on the receding contact width, thus there is almost no effect on the receding contact stresses.

4. The stress intensity factors (SIFs) become more important for this problem since the quarter-planes have sharp edges at which stress concentrations occur. SIFs increase with increasing of the material inhomogeneity μ_h/μ_0. That is, SIFs increase when top of the layer becomes stiffer compared to its bottom.

Appendix A

Expressions of $k_{11}(x_1,t_1)$, $k_{12}(x_1,t_2)$, $k_{21}(x_2,t_1)$, $k_{22}(x_2,t_2)$ and φ_1 appearing in (3.10) and (3.11) are given below:

$$k_{11}(x_1,t_1)=\int_0^\infty \frac{K_{11}(\alpha)-\varphi_1}{\varphi_1}\sin\alpha(t_1-x_1)d\alpha, \tag{A1}$$

$$k_{12}(x_1,t_2)=-2\int_0^\infty \frac{K_{12}(\alpha)}{\varphi_1}\sin\alpha(t_2-x_1)d\alpha, \tag{A2}$$

$$k_{21}(x_2,t_1)=\int_0^\infty \frac{K_{21}(\alpha)}{\varphi_2}\sin\alpha(t_1-x_2)d\alpha, \tag{A3}$$

$$k_{22}(x_2,t_2)=-2\int_0^\infty \frac{K_{22}(\alpha)-\varphi_2}{\varphi_2}\sin\alpha(t_2-x_2)d\alpha, \tag{A4}$$

$$k_{23}(x_2,t_2)=\frac{\varphi_1}{\mu_0}\bar{k}_{23}(x_2,t_2), \tag{A5}$$

where

$$K_{11}(\alpha)=\sum_{j=1}^{4}\alpha m_j e^{n_j h}A_{j1},\quad K_{12}(\alpha)=\sum_{j=1}^{4}\alpha m_j e^{n_j h}A_{j2}, \tag{A6}$$

$$K_{21}(\alpha)=\sum_{j=1}^{4}\alpha m_j A_{j1},\qquad K_{22}(\alpha)=\sum_{j=1}^{4}\alpha m_j A_{j2}. \tag{A7}$$

In above, φ_1 and φ_2 are the singular terms which can be obtained as follows:

$$\varphi_1=\lim_{\alpha\to\infty}K_{11}(\alpha)=-\frac{\kappa+1}{4}e^{\gamma h}, \tag{A8}$$

$$\varphi_2=\lim_{\alpha\to\infty}K_{22}(\alpha)=\frac{\kappa+1}{4}. \tag{A9}$$

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