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Problems of steady vibrations in the theory of Moore–Gibson–Thompson thermoviscoelasticity for materials with voids

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In this paper, the linear theory of Moore–Gibson–Thompson (MGT) thermoviscoelasticity for materials with voids is examined and the basic boundary value problems (BVPs) of steady vibrations are investigated. The governing equations of motion and steady vibrations are formulated. The fundamental solution to the system of steady vibration equations is constructed explicitly using four elementary functions, and its key properties are analyzed. Then, Green's first identity is established and the uniqueness theorems for classical solutions of the associated basic BVPs are proved. The surface and volume potentials are defined, and their essential properties are established. Singular integral operators are introduced, and their symbolic determinants and indices are calculated. Finally, existence theorems for classical solutions of the basic internal and external BVPs are established using the potential method.

Key words: thermoviscoelasticity, materials with voids, steady vibrations, potential method, existence and uniqueness theorems.

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1. Introduction

Thermoviscoelasticity is the branch of continuum mechanics that studies the coupled thermal and viscoelastic behavior of solids, whose deformation depends not only on stress and strain but also on time and temperature. It extends classical viscoelasticity by incorporating thermal effects, taking into account for phenomena such as thermal expansion, temperature-dependent relaxation, and energy dissipation through internal friction and heat generation. Materials such as polymers, asphalt, biological tissues, glasses, and metals at elevated temperatures often exhibit thermoviscoelastic behavior (for details, see Lakes [1], Park and Lakes [2]).

Moreover, materials having small distributed pores may be called porous materials or materials with voids. The intended application of such media may be found in geological and biological materials like rocks and soils and in manufacturing porous materials for which the classical theory of elasticity is not adequate. Nunziato and Cowin [3, 4] developed the theory of elastic materials with voids. This framework was later extended by Ieşan [5], who incorporated Fourier's classical law of heat conduction and presented a corresponding theory of thermoelasticity for materials with voids. Fundamental results and comprehensive reviews of the literature on elastic and thermoelastic materials with voids can be found in the monographs by Ciarletta and Ieşan [6] and Ieşan [7], as well as in a series of papers [8–14] and the references therein.

In parallel, mathematical theories were developed that took into account the viscous properties of materials with voids. Indeed, Cowin [15] presented the theory of viscoelasticity for materials with voids. Then, Ieşan [16] introduced the thermoviscoelasticity theory for the same structure of materials based on the Fourier law of heat conduction. After this, various important problems of the theories of viscoelasticity and thermoviscoelasticity for materials with voids are investigated in the several papers by Bucur [17], Chirita [18], Ieşan and Quintanilla [19], Quintanilla *et al.* [20], Sharma and Kumar [21], Svanadze [22–25], and Tomar *et al.* [26].

On the other hand, it is well known that Fourier's law (parabolic heat equation) predicts infinite thermal signal speed and fails when heat transport shows clear wave-like or nonlocal effects. For this reason, in the second half of the last century, the construction and intensive investigation of generalized thermoelasticity theories based on non-Fourier's law of heat conduction began. The first modification of Fourier's law was proposed by Cattaneo [27] and Vernotte [28]. In this formulation, a single relaxation time is introduced into Fourier's law, which transforms the governing heat equation into a hyperbolic form and leads to the prediction of temperature waves (second sound) propagating at finite speed. Based on the Cattaneo–Vernotte law, Lord and Shulman [29] introduced the first generalized theory of thermoelasticity. Green and Lindsay [30] proposed a theory incorporating two relaxation times (one for the heat flux vector and one for the temperature gradient) in the constitutive equations.

Moreover, in the early 1990s, Green and Naghdi [31–33] proposed three new theories of generalized thermoelasticity, distinguished by their respective laws of heat transfer (Types I, II, and III). A wide historical overview of the non-Fourier heat conduction laws and the generalized thermoelasticity theories is given in the books by Ignaczak and Ostoja-Starzewski [34], Straughan [35], Tzou [36], and in the review papers by Chandrasekharaiah [37, 38], Joseph and Preziosi [39, 40], and Shakeriaski *et al.* [41].

Recently, based on the MGT [42, 43] equation, Quintanilla [44] has introduced the so-called MGT thermoelasticity theory. This theory has found important applications in acoustics, laser-induced heating, and high-frequency thermoelastic wave propagation. Because of this, it has attracted great attention from scientists and has become the subject of intensive research. Important results obtained in this direction are presented in the papers by BAZARRA *et al.* [45, 46], FLOREA and BOBE [47], JANGID *et al.* [48], JANGID and MUKHOPADHYAY [49], MARIN *et al.* [50], QUINTANILLA [51], SINGH and MUKHOPADHYAY [52], SVANADZE [53–56] and the references therein.

In this paper, the linear theory of MGT thermoviscoelasticity for materials with voids is examined, and the basic BVPs associated with steady vibrations are investigated. This model extends IEŞAN'S [16] theory of thermoviscoelasticity for materials with the same structure by replacing the classical Fourier law with the MGT equation for heat propagation.

This paper is organized as follows. Section 2 presents the governing equations of motion and steady vibrations within the linear model of the MGT thermoviscoelasticity for materials with voids. The system of equations of motion is formulated in terms of the displacement vector field, the variation of the pore volume fraction, and the temperature change in the porous material. In Section 3, the fundamental solution to the steady vibration system of equations is explicitly constructed by four elementary functions and its essential properties are established. In Section 4, the basic internal and external BVPs of steady vibrations are formulated, and Green's first identity is obtained. In Section 5, on the basis of this identity, the uniqueness theorems for classical solutions of the aforementioned BVPs are proved. In Section 6, the surface (single-layer and double-layer) and volume potentials are defined, and their basic properties are derived. In Section 7, singular integral operators relevant to the BVPs are introduced and their symbolic determinants and indices are calculated. Finally, the existence theorems for classical solutions to these BVPs are proven using the potential method.

For extensive information on the potential method and its main results in both the classical theories of elasticity and thermoelasticity, as well as in mathematical theories of porous materials, see the monographs by KUPRADZE *et al.* [57] and SVANADZE [58].

2. Governing equations

Within this paper, we assume that an isotropic, homogeneous Kelvin–Voigt porous material occupies a region Ω of the three-dimensional Euclidean space $\mathbb{R}^3$. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of $\mathbb{R}^3$ and let t (≥ 0) denote time.

We use the following standard notations: the Latin subscripts (unless otherwise specified) are understood to range over the integers $(1, 2, 3)$, vectors and matrices are denoted by bold letters, subscripts preceded by a comma indicate partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices imply summation over the range $(1, 2, 3)$, a superposed dot denotes differentiation with respect to time t. In this section, functions and vectors that depend on $\mathbf{x}$ and t are marked with a "hat" symbol.

Let $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ be the displacement vector for the skeleton of the Kelvin–Voigt porous material, $\hat{\varphi}$ be the change of the volume fraction of the pore network from the reference configuration, $\hat{\theta}$ be the temperature measured from some constant absolute temperature $T_0(> 0)$, $\hat{\vartheta}$ be the thermal displacement satisfying the condition [31–33]

$$\dot{\hat{\vartheta}} = \hat{\theta}. \tag{2.1}$$

Following Ieşan [16] and Quintanilla [44], the governing system of field equations of motion in the linear theory of MGT thermoviscoelasticity for materials with voids is composed of the following four sets of equations:

1. *The equations of motion*

$$\hat{t}_{lj,l} = \rho(\ddot{\hat{u}}_j - \hat{F}'_j), \qquad \hat{h}_{l,l} + \hat{g} = \rho_1\ddot{\hat{\varphi}} - \rho\hat{F}'_4, \tag{2.2}$$

where $\hat{t}_{lj}$ is the component of the stress tensor, ρ (> 0) is the reference mass density, $\hat{\mathbf{F}}' = (\hat{F}'_1, \hat{F}'_2, \hat{F}'_3)$ is the body force per unit mass, $\hat{h}_l$ is the components of the equilibrated stress vector, $\hat{g}$ is the equilibrated body force, ρ_1 (> 0) is the equilibrated inertia, $\hat{F}'_4$ is the extrinsic equilibrated body force per unit mass.

2. *The constitutive equations*

$$\begin{aligned} &\hat{t}_{lj} = 2\mu_0\hat{e}_{lj} + \lambda_0\hat{e}_{rr}\delta_{lj} + (b_0\hat{\varphi} - \beta\hat{\theta})\delta_{lj}, \qquad \hat{h}_l = \alpha_0\hat{\varphi}_{,l},\\ &\hat{g} = \nu_0\hat{e}_{rr} - \xi_0\hat{\varphi} + m\hat{\theta}, \qquad \rho\hat{\eta} = a\hat{\theta} + \beta\hat{e}_{rr} + m\hat{\varphi}, \end{aligned} \tag{2.3}$$

where $\hat{\eta}$ is the entropy per unit mass, $a(> 0)$ is the thermal capacity, δ_{lj} is the Kronecker delta,

$$\begin{aligned} \lambda_0 &= \lambda + \lambda^*\frac{\partial}{\partial t}, \qquad \mu_0 = \mu + \mu^*\frac{\partial}{\partial t}, \qquad b_0 = b + b^*\frac{\partial}{\partial t},\\ \alpha_0 &= \alpha + \alpha^*\frac{\partial}{\partial t}, \qquad \nu_0 = b + \gamma^*\frac{\partial}{\partial t}, \qquad \xi_0 = \xi + \xi^*\frac{\partial}{\partial t}, \end{aligned}$$

λ and μ are the Lamé constants, $\beta(\neq 0)$ is the thermal expansion coefficient, b, α, and ξ are the constant parameters corresponding to voids of the porous material, m is the constant thermal parameter, λ^*, μ^*, b^*, α^*, γ^*, and ξ^* are the constant viscoelastic parameters, $\hat{e}_{lj}$ is the component of the strain tensor and defined as

$$\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}). \tag{2.4}$$

3. *The heat transfer equation*

$$\hat{q}_{l,l} = -\rho T_0\dot{\hat{\eta}} + \rho\hat{F}'_5, \tag{2.5}$$

where $\hat{q}_l$ is the component of the fluid flux vector and $\hat{F}'_5$ is the heat source.

4. *The MGT heat conduction equation*

$$\tau\dot{\hat{q}}_l + \hat{q}_l = -(k^*\hat{\vartheta}_{,l} + k\hat{\theta}_{,l}), \tag{2.6}$$

where $k(\geq 0)$ is the thermal conductivity, $k^*(\geq 0)$ is the conductivity rate parameter, and $\tau(\geq 0)$ is the relaxation parameter.

Substituting Eqs. (2.1), (2.3), (2.4), and (2.6) into (2.2) and (2.5) we obtain the following system of motion in the linear theory of MGT thermoviscoelasticity for materials with voids expressed in terms of the displacement vector field $\hat{\mathbf{u}}$, the changes of the volume fraction of pore network $\hat{\varphi}$ and the temperature $\hat{\theta}$:

$$\begin{aligned}
\mu_0\Delta\hat{\mathbf{u}} + (\lambda_0+\mu_0)\nabla\mathrm{div}\,\hat{\mathbf{u}} + b_0\nabla\hat{\varphi} - \beta\nabla\hat{\theta} &= \rho(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}'),\\
(\alpha_0\Delta - \xi_0)\hat{\varphi} - \nu_0\mathrm{div}\,\hat{\mathbf{u}} + m\hat{\theta} &= \rho_1\ddot{\hat{\varphi}} - \rho\hat{F}'_4,\\
k^*\Delta\hat{\theta} + k\Delta\dot{\hat{\theta}} - T_0M(a\dot{\hat{\theta}} + \beta\mathrm{div}\,\dot{\hat{\mathbf{u}}} + \mathbf{m}\dot{\hat{\varphi}}) &= -\rho M\hat{F}'_5,
\end{aligned} \tag{2.7}$$

where Δ is the Laplacian operator, ∇ is the gradient operator, and $M=\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}$.

It should be noted that, for different values of k, k^*, and τ, the system (2.7) yields distinct systems of equations of motion corresponding to various theories of thermoviscoelasticity for materials with voids, each based on a specific law of heat conduction:

(i) The Fourier classical law for $k^* = \tau = 0$, $k > 0$.
(ii) The Cattaneo–Vernotte law for $k^* = 0$, $k > 0$, $\tau > 0$.
(iii) The Green–Naghdi type II law for $k=\tau k^*$ (in particular, $k=\tau=0$), $k^*>0$.
(iv) The Green–Naghdi type III law for $\tau = 0$, $k > 0$, $k^* > 0$.
(v) The MGT law for $k^* > 0$, $\tau > 0$, $k - \tau k^* \neq 0$.

We suppose that $\hat{\mathbf{u}}$, $\hat{\varphi}$, $\hat{\theta}$, and $\hat{F}'_l$ have a harmonic time variation. This means the following

$$\{\hat{\mathbf{u}}, \hat{\varphi}, \hat{\theta}, \hat{F}'_l\}(\mathbf{x}, t) = \mathrm{Re}[\{\mathbf{u}, \varphi, \theta, F'_l\}(\mathbf{x})\, e^{-i\omega t}], \qquad l = 1, 2, \dots, 5.$$

Then from (2.7) we obtain the system of steady vibration equations in the theory of MGT thermoviscoelasticity for materials with voids:

$$\begin{aligned}
(\mu_1\Delta + \rho\omega^2)\mathbf{u} + (\lambda_1+\mu_1)\nabla\mathrm{div}\,\mathbf{u} + b_1\nabla\varphi - \beta\nabla\theta &= \mathbf{F}'',\\
(\alpha_1\Delta + \xi_1)\varphi - \nu_1\mathrm{div}\,\mathbf{u} + m\theta &= F_4,\\
(k_0\Delta + m_1 a)\theta + m_1(\beta\mathrm{div}\,\mathbf{u} + m\varphi) &= F_5,
\end{aligned} \tag{2.8}$$

where ω (> 0) is the oscillation frequency, $\mathbf{F}'' = -\rho\mathbf{F}'$, $F_4 = -\rho F'_4$, $F_5 = \rho(i\omega + \tau\omega^2)F'_5$, and

$$\begin{aligned}
&\lambda_1 = \lambda - i\omega\lambda^*, && \mu_1 = \mu - i\omega\mu^*, && b_1 = b - i\omega b^*,\\
&\alpha_1 = \alpha - i\omega\alpha^*, && \xi_1 = \rho_1\omega^2 - \xi + i\omega\xi^*, && \nu_1 = b - i\omega\gamma^*,\\
&k_0 = k^* - i\omega k, && m_1 = T_0\omega^2(1 - i\omega\tau). &&
\end{aligned} \tag{2.9}$$

We introduce the notation:

$$\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{5\times 5}, \qquad A_{lj} = (\mu_1\Delta + \rho\omega^2)\delta_{lj} + (\lambda_1+\mu_1)\frac{\partial^2}{\partial x_l \partial x_j},$$
$$A_{l4} = b_1\frac{\partial}{\partial x_l}, \qquad A_{l5} = -\beta\frac{\partial}{\partial x_l}, \qquad A_{4l} = -\nu_1\frac{\partial}{\partial x_l}, \qquad A_{44} = \alpha_1\Delta + \xi_1,$$
$$A_{45} = m, \qquad A_{5l} = \beta m_1\frac{\partial}{\partial x_l}, \qquad A_{54} = m m_1, \qquad A_{55} = k_0\Delta + a m_1,$$
$$\mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

Obviously, the system (2.8) we can rewrite in the following form

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.10}$$

where $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ and $\mathbf{F} = (\mathbf{F}'', F_4, F_5)$ are five-component vector functions.

Throughout this paper, we assume that the condition

$$\begin{gathered} \mu^* > 0, \qquad 3\lambda^* + 2\mu^* > 0, \qquad \alpha^* > 0, \qquad k_0 \neq 0, \\ k - \tau k^* \geq 0, \qquad (b^* + \gamma^*)^2 < \frac{4}{3}(3\lambda^* + 2\mu^*)\xi^* \end{gathered} \tag{2.11}$$

is fulfilled.

3. Basic properties of fundamental solution

In this section, the fundamental solution of the system of steady vibration equations (2.8) is introduced and its basic properties are established.

Obviously, the fundamental solution of Eq. (2.10) (the fundamental matrix of operator $\mathbf{A}(\mathbf{D_x})$) is the matrix $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{5\times 5}$ satisfying the following equation in the class of generalized functions

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{5\times 5}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

We introduce the notation:

1)

$$\mathbf{B}(\mathbf{D_x}) = (B_{lj}(\mathbf{D_x}))_{3\times 3} = \begin{pmatrix} \mu_1'\Delta + \rho\omega^2 & -\nu_1\Delta & \beta m_1\Delta \\ b_1 & \alpha_1\Delta + \xi_1 & m m_1 \\ -\beta & m & k_0\Delta + a m_1 \end{pmatrix}_{3\times 3},$$

where $\mu_1' = \lambda_1 + 2\mu_1$.

2)

$$\Lambda_1(\Delta) = \frac{1}{\mu_1'\alpha_1 k_0}\det\mathbf{B}(\Delta) = \prod_{j=1}^{3}(\Delta+\zeta_j^2), \qquad \Lambda_2(\Delta) = \Lambda_1(\Delta)(\Delta+\zeta_4^2),$$

where ζ_1^2, ζ_2^2, and ζ_3^2 are the roots of the equation $\Lambda_1(-\chi) = 0$ (with respect to χ), whereas $\zeta_4^2 = \frac{\rho\omega^2}{\mu_1}$. We suppose that these roots are distinct and $\operatorname{Im}\zeta_l > 0$ $(l = 1, 2, 3, 4)$.

3)

$$\begin{gathered}\mathbf{L}(\mathbf{D_x}) = (L_{lj}(\mathbf{D_x}))_{5\times5}, \qquad L_{lj}(\mathbf{D_x}) = \frac{1}{\mu_1}\Lambda_1(\Delta)\,\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j},\\ L_{lr}(\mathbf{D_x}) = n_{1;r-2}(\Delta)\frac{\partial}{\partial x_l}, \qquad L_{rj}(\mathbf{D_x}) = n_{r-2;1}(\Delta)\frac{\partial}{\partial x_j},\\ L_{rm}(\mathbf{D_x}) = n_{r-2;m-2}(\Delta), \qquad r, m = 4, 5,\end{gathered} \tag{3.1}$$

where

$$\begin{aligned} n_{l1}(\Delta) &= -\frac{1}{\mu_1\mu_1'\alpha_1 k_0}[(\lambda_1+\mu_1)B_{l1}^* - \nu_1 B_{l2}^* + \beta m_1 B_{l3}^*],\\ n_{lj}(\Delta) &= \frac{1}{\mu_1'\alpha k_0}B_{lj}^*, \qquad j = 2, 3,\end{aligned}$$

and B_{lj}^* is the cofactor of element B_{lj} of $\mathbf{B}$.

By direct calculation we get

$$\mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta), \tag{3.2}$$

where

$$\begin{gathered}\mathbf{\Lambda}(\Delta) = (\Lambda_{lj}(\Delta))_{5\times5}, \qquad \Lambda_{11} = \Lambda_{22} = \Lambda_{33} = \Lambda_2,\\ \Lambda_{44} = \Lambda_{55} = \Lambda_1, \qquad \Lambda_{lj} = 0, \qquad l \neq j,\ l, j = 1, 2, \dots, 5.\end{gathered}$$

Let

$$\begin{gathered}\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lj}(\mathbf{x}))_{5\times5}, \quad \Psi_{11}(\mathbf{x}) = \Psi_{22}(\mathbf{x}) = \Psi_{33}(\mathbf{x}) = \sum_{r=1}^{4}\eta_{2r}\psi^{(r)}(\mathbf{x}),\\ \Psi_{44}(\mathbf{x}) = \Psi_{55}(\mathbf{x}) = \sum_{r=1}^{3}\eta_{1r}\psi^{(r)}(\mathbf{x}), \quad \Psi_{lj}(\mathbf{x}) = 0, \quad l \neq j,\ l, j = 1, 2, 3, 4,\end{gathered} \tag{3.3}$$

where

$$\begin{aligned}\eta_{1r} &= \prod_{l=1,\, l\neq r}^{4}(\zeta_l^2 - \zeta_r^2)^{-1}, \qquad \eta_{2m} = \prod_{l=1,\, l\neq m}^{3}(\zeta_l^2 - \zeta_m^2)^{-1},\\ \psi^{(r)}(\mathbf{x}) &= -\frac{e^{i\zeta_r|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \qquad r = 1, 2, 3, 4.\end{aligned}$$

Obviously, the matrix $\mathbf{\Psi}(\mathbf{x})$ is the fundamental solution of the operator $\mathbf{\Lambda}(\Delta)$, that is,

$$\mathbf{\Lambda}(\Delta)\mathbf{\Psi}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{3.4}$$

where $\mathbf{x} \in \mathbb{R}^3$.

We have the following theorem.

THEOREM 1. *The matrix $\mathbf{\Gamma}(\mathbf{x})$, defined by*

$$\mathbf{\Gamma}(\mathbf{x}) = \mathbf{L}(\mathbf{D_x})\mathbf{\Psi}(\mathbf{x}), \tag{3.5}$$

is the fundamental solution of the system (2.8), *where the matrices $\mathbf{L}(\mathbf{D_x})$ and $\mathbf{\Psi}(\mathbf{x})$ are given by* (3.1) *and* (3.3), *respectively.*

Proof. Using identities (3.2) and (3.4) from (3.5) it follows that

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{A}(\mathbf{D_x})\mathbf{L}(\mathbf{D_x})\mathbf{\Psi}(\mathbf{x}) = \mathbf{\Lambda}(\Delta)\mathbf{\Psi}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Hence, $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$. □

It is worth noting that the matrix $\mathbf{\Gamma}(\mathbf{x})$ is presented explicitly with the help of four elementary functions $\psi^{(j)}$ $(j = 1, 2, 3, 4)$.

Based on Theorem 1, we obtain the following essential properties of the matrix $\mathbf{\Gamma}(\mathbf{x})$.

THEOREM 2. *Each column of the matrix $\mathbf{\Gamma}(\mathbf{x})$ is a solution of homogeneous equation $\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ at every point $\mathbf{x} \in \mathbb{R}^3$ except the origin of $\mathbb{R}^3$.*

THEOREM 3. *In the neighbourhood of the origin of $\mathbb{R}^3$ there holds*

$$\begin{array}{lll} \Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \Gamma_{44}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \Gamma_{55}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \\ \Gamma_{lr}(\mathbf{x}) = O(1), & \Gamma_{rl}(\mathbf{x}) = O(1), & \Gamma_{45}(\mathbf{x}) = O(1), \\ \Gamma_{54}(\mathbf{x}) = O(1), & \quad r = 4, 5. & \end{array}$$

THEOREM 4. *Let the matrix differential operator $\mathbf{A}^{(0)}(\mathbf{D_x})$ defined by*

$$\mathbf{A}^{(0)}(\mathbf{D_x}) = (A^{(0)}_{lj}(\mathbf{D_x}))_{5\times 5}, \qquad A^{(0)}_{lj}(\mathbf{D_x}) = \mu_1\Delta\delta_{lj} + (\lambda_1 + \mu_1)\frac{\partial^2}{\partial x_l \partial x_j},$$
$$A^{(0)}_{44}(\mathbf{D_x}) = \alpha_1\Delta, \qquad A^{(0)}_{55}(\mathbf{D_x}) = k_0\Delta, \qquad A^{(0)}_{lr} = A^{(0)}_{rl} = A^{(0)}_{45} = A^{(0)}_{54} = 0.$$

Then $\mathbf{\Gamma}^{(0)}(\mathbf{x}) = (\Gamma^{(0)}_{lj}(\mathbf{x}))_{5\times 5}$ is the fundamental matrix of the operator $\mathbf{A}^{(0)}(\mathbf{D_x})$ with the following properties

$$\begin{array}{lll} \Gamma^{(0)}_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \Gamma^{(0)}_{44}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & \Gamma^{(0)}_{55}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \\ \multicolumn{3}{l}{\Gamma^{(0)}_{lr}(\mathbf{x}) = \Gamma^{(0)}_{rl}(\mathbf{x}) = \Gamma^{(0)}_{45}(\mathbf{x}) = \Gamma^{(0)}_{54}(\mathbf{x}) = 0} \end{array}$$

in the neighbourhood of the origin of $\mathbb{R}^3$, *where:*

$$\Gamma^{(0)}_{lj}(\mathbf{x}) = -\frac{\lambda_1+3\mu_1}{8\pi\mu_1\mu_1'}\,\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{\lambda_1+\mu_1}{8\pi\mu_1\mu_1'}\,\frac{x_l x_j}{|\mathbf{x}|^3}, \qquad \Gamma^{(0)}_{44}(\mathbf{x}) = \frac{1}{\alpha_1}\psi^{(0)}(\mathbf{x}),$$
$$\Gamma^{(0)}_{55}(\mathbf{x}) = \frac{1}{k_0}\psi^{(0)}(\mathbf{x}), \qquad \psi^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \qquad r=4,5.$$

THEOREM 5. *In the neighbourhood of the origin of* $\mathbb{R}^3$ *there holds*

$$\Gamma_{lj}(\mathbf{x}) - \Gamma^{(0)}_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \qquad l,j=1,2,\dots,5.$$

Hence, in view of Theorem 5, the singular part of $\mathbf{\Gamma}(\mathbf{x})$ in the neighbourhood of the origin of $\mathbb{R}^3$ is given by the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$.

4. Basic boundary value problems and Green's identity

At the beginning, we give a definition of the class of regular vector functions. Then the basic BVPs of steady vibrations in the theory under consideration are formulated, and finally, Green's first identity of this theory is established.

Let Ω^+ be a finite domain in $\mathbb{R}^3$ with the surface S, $S \in C^{2,\nu}$, $0<\nu\le 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$.

A vector function $\mathbf{U} = (U_1, U_2, \dots, U_5)$ is called *regular* in Ω^- (or Ω^+) if:

(i) $U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ (or $U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$),

(ii)

$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad U_{l,j}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \tag{4.1}$$

for $|\mathbf{x}| \gg 1$, where $l=1,2,\dots,5$.

We introduce the matrix differential operator $\mathbf{R}(\mathbf{D}_\mathbf{x},\mathbf{n})$ as:

$$\begin{aligned} &\mathbf{R}(\mathbf{D}_\mathbf{x},\mathbf{n}) = (R_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}))_{5\times5}, \quad R_{lj} = \mu_1\delta_{lj}\frac{\partial}{\partial\mathbf{n}} + \mu_1 n_j\frac{\partial}{\partial x_l} + \lambda_1 n_l\frac{\partial}{\partial x_j}, \\ &\qquad R_{l4} = b_1 n_l, \qquad R_{l5} = -\beta n_l, \qquad R_{44} = \alpha_1\frac{\partial}{\partial\mathbf{n}}, \qquad R_{55} = k_0\frac{\partial}{\partial\mathbf{n}}, \\ &\qquad R_{4l} = R_{45} = R_{5l} = R_{54} = 0, \end{aligned} \tag{4.2}$$

where $\mathbf{n}(\mathbf{z})$ denotes the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$, $\mathbf{n} = (n_1,n_2,n_3)$, $\frac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the linear theory of MGT thermoeviscolasticity for materials with voids can be formulated as follows.

Find a regular solution $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ to Eq. (2.10) for $\mathbf{x} \in \Omega^+$ satisfying the boundary conditions:

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.3}$$

and

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{4.4}$$

in the internal BVPs $(I)^+_{\mathbf{F},\mathbf{f}}$ and $(II)^+_{\mathbf{F},\mathbf{f}}$, respectively.

Find a regular solution $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ to Eq. (2.10) for $\mathbf{x} \in \Omega^-$ satisfying the boundary conditions:

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{4.5}$$

and

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{4.6}$$

in the external BVPs $(I)^-_{\mathbf{F},\mathbf{f}}$ and $(II)^-_{\mathbf{F},\mathbf{f}}$, respectively. Here $\mathbf{F}$ and $\mathbf{f}$ are given five-component vector functions, and $\operatorname{supp}\mathbf{F}$ is a finite subset of Ω^-.

In the following, the scalar product of two vectors $\mathbf{v} = (v_1, v_2, \dots, v_l)$ and $\mathbf{v}' = (v_1', v_2', \dots, v_l')$ is denoted by $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{l} v_j \overline{v_j'}$, where $\overline{v_j'}$ is the complex conjugate of v_j'.

We introduce the notation:

$$\begin{aligned} W^{(0)}(\mathbf{u}, \mathbf{u}') &= \frac{1}{3}(3\lambda_1 + 2\mu_1)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{u}'} \\ &\quad + \frac{\mu_1}{2} \sum_{l,j=1;\, l\neq j}^{3} (u_{j,l} + u_{l,j})(\overline{u'_{j,l}} + \overline{u'_{l,j}}) \\ &\quad + \frac{\mu_1}{3} \sum_{l,j=1}^{3} \left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \overline{u_l'}}{\partial x_l} - \frac{\partial \overline{u_j'}}{\partial x_j}\right), \end{aligned} \tag{4.7}$$

$$W^{(1)}(\mathbf{U}, \mathbf{u}') = W^{(0)}(\mathbf{u}, \mathbf{u}') - \rho\omega^2 \mathbf{u} \cdot \mathbf{u}' + (b_1\varphi - \beta\theta)\operatorname{div}\overline{\mathbf{u}'},$$

$$W^{(2)}(\mathbf{U}, \varphi') = \alpha_1 \nabla\varphi \cdot \nabla\varphi' - (\xi_1\varphi - \nu_1 \operatorname{div}\mathbf{u} + m\theta)\overline{\varphi'},$$

$$W^{(3)}(\mathbf{U}, \theta') = k_0 \nabla\theta \cdot \nabla\theta' - m_1(a\theta + \beta\operatorname{div}\mathbf{u} + m\varphi)\overline{\theta'},$$

where u_l', φ' and θ' are functions on $\mathbb{R}^3$, $\mathbf{u}' = (u_1', u_2', u_3')$.

With the help of (4.1) and (4.7), we have the following result.

LEMMA 1. *If* $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi', \theta')$ *are five-component regular vectors in* $\Omega^\pm$, *then*

$$\int_{\Omega^\pm}[\mathbf{A}^{(1)}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x})+W^{(1)}(\mathbf{U},\mathbf{u}')]\,d\mathbf{x}=\pm\int_S\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_\mathbf{z}S,$$

$$\int_{\Omega^\pm}[\mathbf{A}^{(2)}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x})\overline{\varphi'(\mathbf{x})}+W^{(2)}(\mathbf{U},\varphi')]\,d\mathbf{x}=\pm\alpha_1\int_S\frac{\partial\varphi}{\partial\mathbf{n}}\,\overline{\varphi'}\,d_\mathbf{z}S, \tag{4.8}$$

$$\int_{\Omega^\pm}[\mathbf{A}^{(3)}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x})\,\overline{\theta'(\mathbf{x})}+W^{(3)}(\mathbf{U},\theta')]\,d\mathbf{x}=\pm k_0\int_S\frac{\partial\theta}{\partial\mathbf{n}}\,\overline{\theta'}\,d_\mathbf{z}S,$$

where

$$\mathbf{A}^{(1)}(\mathbf{D}_\mathbf{x})=(A^{(1)}_{lj}(\mathbf{D}_\mathbf{x}))_{3\times5},\qquad A^{(1)}_{lj}(\mathbf{D}_\mathbf{x})=A_{lj}(\mathbf{D}_\mathbf{x}),$$
$$\mathbf{A}^{(r)}(\mathbf{D}_\mathbf{x})=(A^{(r)}_{1j}(\mathbf{D}_\mathbf{x}))_{1\times5},\qquad A^{(r)}_{1j}(\mathbf{D}_\mathbf{x})=A_{r+2;j}(\mathbf{D}_\mathbf{x}),$$
$$\mathbf{R}^{(1)}(\mathbf{D}_\mathbf{x},\mathbf{n})=(R^{(1)}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}))_{3\times5},\qquad R^{(1)}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n})=R_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}),$$
$$j=1,2,\dots,5,\;\; r=2,3.$$

On the basis of Lemma 1 we can easily verify the following theorem.

Theorem 6. *If* $\mathbf{U}=(\mathbf{u},\varphi,\theta)$ *and* $\mathbf{U}'=(\mathbf{u}',\varphi',\theta')$ *are five-component regular vectors in* $\Omega^\pm$, *then*

$$\int_{\Omega^\pm}[\mathbf{A}(\mathbf{D}_\mathbf{x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} =\pm\int_S\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_\mathbf{z}S, \tag{4.9}$$

where the operator $\mathbf{R}$ *is given by* (4.2) *and*

$$W(\mathbf{U},\mathbf{U}')=W^{(1)}(\mathbf{U},\mathbf{u}')+W^{(2)}(\mathbf{U},\varphi')+W^{(3)}(\mathbf{U},\theta').$$

The relation (4.9) is Green's first identity in the linear theory of MGT thermoeviscolasticity for materials with voids.

Obviously, from (4.7) it follows that:

$$\begin{aligned}
W^{(0)}(\mathbf{u},\mathbf{u})&=\frac{1}{3}(3\lambda_1+2\mu_1)\,|\mathrm{div}\,\mathbf{u}|^2+\frac{\mu_1}{2}\sum_{l,j=1;\,l\neq j}^{3}|u_{j,l}+u_{l,j}|^2\\
&\quad+\frac{\mu_1}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2,\\
W^{(1)}(\mathbf{U},\mathbf{u})&=W^{(0)}(\mathbf{u},\mathbf{u})-\rho\omega^2|\mathbf{u}|^2+(b_1\varphi-\beta\theta)\,\mathrm{div}\,\overline{\mathbf{u}},\\
W^{(2)}(\mathbf{U},\varphi)&=\alpha_1|\nabla\varphi|^2-\xi_1|\varphi|^2+(\nu_1\,\mathrm{div}\,\mathbf{u}-m\theta)\overline{\varphi},\\
W^{(3)}(\mathbf{U},\theta)&=k_0|\nabla\theta|^2-am_1|\theta|^2-m_1(\beta\,\mathrm{div}\,\mathbf{u}+m\varphi)\overline{\theta}.
\end{aligned} \tag{4.10}$$

In view of (2.9) from (4.10) we have:

$$
\begin{aligned}
\operatorname{Im} W^{(1)}(\mathbf{U},\mathbf{u}) &= -\omega W^*(\mathbf{u}) + b \operatorname{Im}(\varphi \operatorname{div} \overline{\mathbf{u}}) - \omega b^* \operatorname{Re}(\varphi \operatorname{div} \overline{\mathbf{u}}) \\
&\quad - \beta \operatorname{Im}(\theta \operatorname{div} \overline{\mathbf{u}}), \\
\operatorname{Im} W^{(2)}(\mathbf{U},\varphi) &= -\omega\alpha^* |\nabla\varphi|^2 - \omega\xi^* |\varphi|^2 - m \operatorname{Im}(\theta\overline{\varphi}) + b \operatorname{Im}(\operatorname{div} \mathbf{u}\overline{\varphi}) \\
&\quad - \omega\gamma^* \operatorname{Re}(\operatorname{div} \mathbf{u}\overline{\varphi}), \\
\operatorname{Re} W^{(3)}(\mathbf{U},\theta) &= k^* |\nabla\theta|^2 - a m_2 |\theta|^2 - \beta m_2 \operatorname{Re}(\operatorname{div} \mathbf{u}\overline{\theta}) \\
&\quad - \omega\tau\beta m_2 \operatorname{Im}(\operatorname{div} \mathbf{u}\overline{\theta}) - m m_2 \operatorname{Re}(\varphi\overline{\theta}) \\
&\quad - \omega\tau m m_2 \operatorname{Im}(\varphi\overline{\theta}), \\
\operatorname{Im} W^{(3)}(\mathbf{U},\theta) &= -\omega k |\nabla\theta|^2 + \omega\tau a m_2 |\theta|^2 - \beta m_2 \operatorname{Im}(\operatorname{div} \mathbf{u}\overline{\theta}) \\
&\quad + \omega\tau\beta m_2 \operatorname{Re}(\operatorname{div} \mathbf{u}\overline{\theta}) - m m_2 \operatorname{Im}(\varphi\overline{\theta}) \\
&\quad + \omega\tau m m_2 \operatorname{Re}(\varphi\overline{\theta}),
\end{aligned}
\tag{4.11}
$$

where $m_2 = T_0\omega^2$ and

$$
\begin{aligned}
W^*(\mathbf{u}) &= \frac{1}{3}(3\lambda^* + 2\mu^*)\,|\operatorname{div}\, \mathbf{u}|^2 + \frac{\mu^*}{2} \sum_{l,j=1;\, l\neq j}^{3} |u_{j,l} + u_{l,j}|^2 \\
&\quad + \frac{\mu^*}{3} \sum_{l,j=1}^{3} \left| \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j} \right|^2 .
\end{aligned}
\tag{4.12}
$$

At a glance, by virtue of (4.12) from (4.11) it can be verified that:

$$
\begin{aligned}
&\operatorname{Im} W^{(1)}(\mathbf{U},\mathbf{u}) + \operatorname{Im} W^{(2)}(\mathbf{U},\varphi) \\
&\quad = -\omega W^*(\mathbf{u}) - (b^* + \gamma^*) \operatorname{Re}(\operatorname{div} \mathbf{u}\overline{\varphi}) \\
&\qquad - \beta \operatorname{Im}(\theta \operatorname{div} \overline{\mathbf{u}}) - \omega\alpha^* |\nabla\varphi|^2 - \omega\xi^* |\varphi|^2 - m \operatorname{Im}(\theta\overline{\varphi}) \\
&\quad = -\omega W_0^*(\mathbf{u}) - \omega W^{(4)}(\mathbf{u},\varphi) - \omega\alpha^* |\nabla\varphi|^2 - \beta \operatorname{Im}(\theta \operatorname{div} \overline{\mathbf{u}}) - m \operatorname{Im}(\theta\overline{\varphi}), \\
&\omega\tau \operatorname{Re} W^{(3)}(\mathbf{U},\theta) + \operatorname{Im} W^{(3)}(\mathbf{U},\theta) \\
&\quad = -\omega k' |\nabla\theta|^2 - \beta\tau_0 m_2 \operatorname{Im}(\operatorname{div} \mathbf{u}\overline{\theta}) - m\tau_0 m_2 \operatorname{Im}(\varphi\overline{\theta}),
\end{aligned}
\tag{4.13}
$$

where $k' = k - \tau k^*$, $\tau_0 = 1 + \omega^2\tau^2$, and

$$
\begin{aligned}
W_0^*(\mathbf{u}) &= \frac{\mu^*}{2} \sum_{l,j=1;\, l\neq j}^{3} |u_{j,l} + u_{l,j}|^2 + \frac{\mu^*}{3} \sum_{l,j=1}^{3} \left| \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j} \right|^2 \geq 0, \\
W^{(4)}(\mathbf{u},\varphi) &= \frac{1}{3}(3\lambda^* + 2\mu^*)\,|\operatorname{div}\, \mathbf{u}|^2 \\
&\quad + (b^* + \gamma^*) \operatorname{Re}(\operatorname{div} \mathbf{u}\overline{\varphi}) + \xi^* |\varphi|^2 \geq 0.
\end{aligned}
\tag{4.14}
$$

Consequently, in view of (4.14) from (4.13) we deduce that

$$\begin{aligned} & -m_2\tau_0[\operatorname{Im} W^{(1)}(\mathbf{U},\mathbf{u}) + \operatorname{Im} W^{(2)}(\mathbf{U},\varphi)] \\ & \qquad - \omega\tau \operatorname{Re} W^{(3)}(\mathbf{U},\theta) - \operatorname{Im} W^{(3)}(\mathbf{U},\theta) \\ & = \omega m_2\tau_0[W_0^*(\mathbf{u}) + W^{(4)}(\mathbf{u},\varphi) + \alpha^*|\nabla\varphi|^2] + \omega k'|\nabla\theta|^2 \ge 0. \end{aligned} \tag{4.15}$$

5. Uniqueness theorems

In this section, the uniqueness theorems for classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ are proved.

We have the following uniqueness theorems.

THEOREM 7. *The internal BVP $(K)^{+}_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution, where $K = I, II$.*

Proof. We suppose that there are two regular solutions of the BVP $(K)^{+}_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(K)^{+}_{\mathbf{0},\mathbf{0}}$, i.e., $\mathbf{U}$ is a regular solution in Ω^+ of the system of homogeneous equations:

$$\begin{aligned} (\mu_1\Delta + \rho\omega^2)\mathbf{u} + (\lambda_1 + \mu_1)\nabla \operatorname{div} \mathbf{u} + b_1\nabla\varphi - \beta\nabla\theta &= \mathbf{0}, \\ (\alpha_1\Delta + \xi_1)\varphi - \nu_1 \operatorname{div} \mathbf{u} + m\theta &= 0, \\ (k_0\Delta + m_1 a)\theta + m_1(\beta \operatorname{div} \mathbf{u} + m\varphi) &= 0, \end{aligned} \tag{5.1}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S \tag{5.2}$$

in the internal *Problem* $(I)^{+}_{\mathbf{0},\mathbf{0}}$, and

$$\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \tag{5.3}$$

in the internal *Problem* $(II)^{+}_{\mathbf{0},\mathbf{0}}$, where the operator $\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))$ is defined by (4.2).

In view of the relations (5.1)–(5.3), from (4.8) for $\mathbf{U}' = \mathbf{U}$ we deduce that:

$$\int_{\Omega^+} W^{(1)}(\mathbf{U},\mathbf{u})\,d\mathbf{x} = 0, \quad \int_{\Omega^+} W^{(2)}(\mathbf{U},\varphi)\,d\mathbf{x} = 0, \quad \int_{\Omega^+} W^{(3)}(\mathbf{U},\theta)\,d\mathbf{x} = 0. \tag{5.4}$$

Now, from (5.4) we can easily verify that

$$\begin{aligned} & m_0\tau_0 \int_{\Omega^+} [\operatorname{Im} W^{(1)}(\mathbf{U},\mathbf{u}) + \operatorname{Im} W^{(2)}(\mathbf{U},\varphi)]\,d\mathbf{x} \\ & \qquad + \int_{\Omega^+} [\omega\tau \operatorname{Re} W^{(3)}(\mathbf{U},\theta) + \operatorname{Im} W^{(3)}(\mathbf{U},\theta)]\,d\mathbf{x} = 0. \end{aligned} \tag{5.5}$$

Taking into account the identity (4.15) from (5.5) we get

$$\int\limits_{\Omega^+} \{\omega m_2 \tau_0 [W_0^*(\mathbf{u}) + W^{(4)}(\mathbf{u},\varphi) + \alpha^* |\nabla\varphi|^2] + \omega k' |\nabla\theta|^2\}\, d\mathbf{x} = 0. \tag{5.6}$$

By virtue of the assumption (2.11) and the relations of (4.14) from (5.6) we obtain

$$W_0^*(\mathbf{u}) = 0, \qquad W^{(4)}(\mathbf{u},\varphi) = 0. \tag{5.7}$$

On the basis of (5.7) from (4.14) it follows that:

$$u_{l,j}(\mathbf{x}) + u_{j,l}(\mathbf{x}) = 0, \quad \frac{\partial u_l(\mathbf{x})}{\partial x_l} - \frac{\partial u_j(\mathbf{x})}{\partial x_j} = 0, \quad \operatorname{div} \mathbf{u}(\mathbf{x}) = 0, \quad l \neq j \tag{5.8}$$

and

$$\varphi(\mathbf{x}) = 0 \tag{5.9}$$

for $\mathbf{x} \in \Omega^+$. Clearly, by relation of (5.8) from (4.10) we get $W^{(0)}(\mathbf{u},\mathbf{u}) = 0$ and consequently, we can write

$$W^{(1)}(\mathbf{U},\mathbf{u}) = -\rho\omega^2 |\mathbf{u}|^2. \tag{5.10}$$

Substitution of (5.10) into the first relation of (5.4) yields

$$\mathbf{u}(\mathbf{x}) = \mathbf{0} \tag{5.11}$$

for $\mathbf{x} \in \Omega^+$.

Afterwards, on the basis of (5.9) and (5.11) from the first equation of (5.1) we obtain $\nabla\theta(\mathbf{x}) = \mathbf{0}$, i.e.,

$$\theta(\mathbf{x}) = \text{const} \tag{5.12}$$

for $\mathbf{x} \in \Omega^+$. Now, taking into account the relations (5.9), (5.11), and (5.12) from the last equation of (5.1) we deduce that $am_1\theta(\mathbf{x}) = 0$, i.e.,

$$\theta(\mathbf{x}) = 0 \tag{5.13}$$

for $\mathbf{x} \in \Omega^+$. Therefore, on the basis of (5.9), (5.11), and (5.13) we have $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^+$ and we have desired result. □

THEOREM 8. *The external BVP $(K)^-_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution, where $K = I, II$.*

Proof. We suppose that there are two regular solutions of the external BVP $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. Then their difference $\mathbf{U}$ is a regular solution of the homogeneous BVP $(K)^-_{\mathbf{0},\mathbf{0}}$, that is, $\mathbf{U}$ is a regular solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\mathbf{U}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^- \tag{5.14}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for } \mathbf{z} \in S \tag{5.15}$$

in the external *Problem* $(I)^-_{\mathbf{0},\mathbf{0}}$, and

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{5.16}$$

in the external *Problem* $(II)^-_{\mathbf{0},\mathbf{0}}$.

In view of the relations of (5.14)–(5.16), from (4.8) for $\mathbf{U}' = \mathbf{U}$ we deduce that:

$$\int\limits_{\Omega^-} W^{(1)}(\mathbf{U}, \mathbf{u})\, d\mathbf{x} = 0, \quad \int\limits_{\Omega^-} W^{(2)}(\mathbf{U}, \varphi)\, d\mathbf{x} = 0, \quad \int\limits_{\Omega^-} W^{(3)}(\mathbf{U}, \theta)\, d\mathbf{x} = 0. \tag{5.17}$$

Obviously, from (5.17) we have

$$\int\limits_{\Omega^-} \left\{\omega m_2 \tau_0 [W_0^*(\mathbf{u}) + W^{(4)}(\mathbf{u}, \varphi) + \alpha^* |\nabla\varphi|^2] + \omega k' |\nabla\theta|^2\right\} d\mathbf{x} = 0. \tag{5.18}$$

As in Theorem 7, similar reasoning applied to Eq. (5.18) yields relations (5.9), (5.11), and (5.13) for $\mathbf{x} \in \Omega^-$. Therefore, $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^-$ and we have desired result. □

6. Basic properties of potentials

In this section, the surface and volume potentials of the theory of MGT thermoviscoelasticity for materials with voids are defined and their essential properties are established.

Let us introduce the following potentials:

(i) $\mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ is the single-layer potential,

(ii) $\mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) = \int_S [\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}}, \mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x} - \mathbf{y})]^\top \mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ is the double-layer potential, and

(iii) $\mathbf{Q}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^\pm) = \int_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\boldsymbol{\phi}(\mathbf{y})\, d\mathbf{y}$ is the volume potential, where $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ and given by (3.5), $\mathbf{g}$ and $\boldsymbol{\phi}$ are five-component vector functions, the matrix differential operator $\tilde{\mathbf{R}}$ defined as:

$$\tilde{\mathbf{R}}(\mathbf{D_x},\mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D_x},\mathbf{n}))_{5\times 5}, \quad \tilde{R}_{lj}(\mathbf{D_x},\mathbf{n}) = \mu_1\delta_{lj}\frac{\partial}{\partial \mathbf{n}} + \mu_1 n_j\frac{\partial}{\partial x_l} + \lambda_1 n_l\frac{\partial}{\partial x_j},$$

$$\tilde{R}_{l4}(\mathbf{D_x},\mathbf{n}) = \nu_1 n_l, \quad \tilde{R}_{l5}(\mathbf{D_x},\mathbf{n}) = -\beta m_1 n_l, \quad \tilde{R}_{rm}(\mathbf{D_x},\mathbf{n}) = R_{rm}(\mathbf{D_x},\mathbf{n}),$$
$$r=4,5,\ m=1,2,\dots,5.$$

Using Theorems 1 to 5, we derive the following four consequences, which present the fundamental properties of these potentials.

THEOREM 9. *If* $S\in C^{2,\nu}$, $\mathbf{g}\in C^{1,\nu'}(S)$, $0<\nu'<\nu\le 1$, *then:*

(a) $\mathbf{Q}^{(1)}(\cdot,\mathbf{g})\in C^{0,\nu'}(\mathbb{R}^3)\cap C^{2,\nu'}(\overline{\Omega^\pm})\cap C^\infty(\Omega^\pm)$,

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g})=\mathbf{0}$,

(c)

$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}), \tag{6.1}$$

(d) $\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})$

is a singular integral, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^\pm$.

THEOREM 10. *If* $S\in C^{2,\nu}$, $\mathbf{g}\in C^{1,\nu'}(S)$, $0<\nu'<\nu\le 1$, *then:*

(a) $\mathbf{Q}^{(2)}(\cdot,\mathbf{g})\in C^{1,\nu'}(\overline{\Omega^\pm})\cap C^\infty(\Omega^\pm)$,

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g})=\mathbf{0}$,

(c)

$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^\pm = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \tag{6.2}$$

(d) $\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where* $\mathbf{z}\in S$,

(e) $\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^+ = \{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^-$,

where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^\pm$.

THEOREM 11. *If* $S\in C^{1,\nu}$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^+)$, $0<\nu'<\nu\le 1$, *then:*

(a) $\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^+)\cap C^{2,\nu'}(\overline{\Omega_0^+})$,

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^+)=\boldsymbol{\phi}(\mathbf{x})$,

where Ω_0^+ *is a domain in* $\mathbb{R}^3$, $\overline{\Omega_0^+}\subset\Omega^+$, *and* $\mathbf{x}\in\Omega^+$.

THEOREM 12. *If* $S\in C^{1,\nu}$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^-)$, $\operatorname{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $0<\nu'<\nu\le 1$, *then:*

(a) $\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,\nu'}(\overline{\Omega_0^-})$,

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-)=\boldsymbol{\phi}(\mathbf{x})$,

where Ω *is a finite domain in* $\mathbb{R}^3$, $\overline{\Omega_0^-}\subset\Omega^-$, *and* $\mathbf{x}\in\Omega^-$.

7. Existence theorems

In this section, we first establish the properties of certain singular integral operators. These properties are then used to prove the existence theorems for the classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$.

We note that the basic definitions and theorems of the theory of singular integral operators are given in the book by KUPRADZE *et al.* [57].

For the subsequent analysis, we require the following matrix singular integral operators:

$$
\begin{aligned}
\mathcal{L}^{(1)}\,\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(2)}\,\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(3)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(4)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}_\chi\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z})+\chi\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})
\end{aligned}
\tag{7.1}
$$

for $\mathbf{z}\in S$, where χ is a complex number. The symbol of the operator $\mathcal{L}^{(j)}$ $(j=1,2,3,4)$ we denote by $\boldsymbol{\Phi}^{(j)}=(\Phi^{(j)}_{lm})_{5\times5}$. Keeping in mind Theorem 5 and the assumption (2.11) from (7.1) we find

$$
\begin{aligned}
&\det\boldsymbol{\Phi}^{(1)}=-\det\boldsymbol{\Phi}^{(2)}=-\det\boldsymbol{\Phi}^{(3)}=\det\boldsymbol{\Phi}^{(4)}\\
&\quad=\left(-\frac{1}{2}\right)^5\left[\frac{\mu_1^2}{(\lambda_1+2\mu_1)^2}-1\right]=\frac{(\lambda_1+\mu_1)(\lambda_1+3\mu_1)}{32(\lambda_1+2\mu_1)^2}\neq0,
\end{aligned}
\tag{7.2}
$$

i.e., the operator $\mathcal{L}^{(j)}$ $(j=1,2,3,4)$ is of the normal type.

Let $\boldsymbol{\Phi}_\chi$ be the symbol of the integral operator $\mathcal{L}_\chi$ and $\operatorname{ind}\mathcal{L}_\chi$ be the index of this operator. From (7.1) we can write

$$
\det\boldsymbol{\Phi}_\chi=-\frac{\mu_1^2\chi^2-(\lambda_1+2\mu_1)^2}{32(\lambda_1+2\mu_1)^2}
$$

and consequently, only at two points χ_1 and χ_2 of the complex plane we have $\det\boldsymbol{\Phi}_\chi=0$.

It is easily verified that in view of (7.2) and $\det \mathbf{\Phi}_1 = \det \mathbf{\Phi}^{(1)}$ we get $\chi_j \neq 1$ $(j = 1, 2)$ and

$$\operatorname{ind} \mathcal{L}_1 = \operatorname{ind} \mathcal{L}^{(1)} = \operatorname{ind} \mathcal{L}_0 = 0.$$

Obviously, from (7.1) we obtain

$$\operatorname{ind} \mathcal{L}^{(2)} = -\operatorname{ind} \mathcal{L}^{(3)} = 0, \qquad \operatorname{ind} \mathcal{L}^{(4)} = -\operatorname{ind} \mathcal{L}^{(1)} = 0.$$

Thus, the operator $\mathcal{L}^{(j)}$ is of the normal type with an index equal to zero, and therefore, Noether's theorems are valid for the singular integral operator $\mathcal{L}^{(j)}$ $(j = 1, 2, 3, 4)$.

We recall that the volume potential $\mathbf{Q}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^{\pm})$ is a regular solution of Eq. (2.10), where $\mathbf{F} \in C^{0,\nu'}(\Omega^{\pm})$, $0 < \nu' \leq 1$; $\operatorname{supp} \mathbf{F}$ is a finite domain in Ω^- (see Theorems 11 and 12). On account of this reason we establish the existence theorems for the BVPs $(I)^{\pm}_{\mathbf{0},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{0},\mathbf{f}}$.

THEOREM 13. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \leq 1$, then a regular solution of the BVP $(I)^{+}_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the double-layer potential*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \textit{for } \mathbf{x} \in \Omega^+, \tag{7.3}$$

where $\mathbf{g}$ is a solution of the singular integral equation

$$\mathcal{L}^{(1)} \mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \textit{for } \mathbf{z} \in S \tag{7.4}$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Proof. We seek a regular solution to the BVP $(I)^{+}_{\mathbf{0},\mathbf{f}}$ in the form of the double-layer potential (7.3), where $\mathbf{g}$ is the density of the potential and represents the unknown five-component vector function. By Theorem 10, the vector function $\mathbf{U}$ satisfies the homogeneous equation (5.1).

Moreover, in view of Eqs. (4.3), (6.2), and (7.1), substitution into (7.3) yields Eq. (7.4) for the unknown vector function $\mathbf{g}$. It remains to prove that Eq. (7.4) is solvable for an arbitrary vector function $\mathbf{f}$.

We now consider the adjoint homogeneous equation corresponding to Eq. (7.4), which takes the following form:

$$\mathcal{L}^{(4)} \mathbf{h}_0(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S, \tag{7.5}$$

where $\mathbf{h}_0$ is the required five-component vector function. Let $\mathbf{h}_0$ be a solution of the homogeneous Eq. (7.5). In view of Theorem 9 and Eq. (7.5), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the BVP $(II)^{-}_{\mathbf{0},\mathbf{0}}$. By Theorem 8, the BVP $(II)^{-}_{\mathbf{0},\mathbf{0}}$ admits only the trivial solution, and therefore we can write:

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^-. \tag{7.6}$$

By Theorem 9 and Eq. (7.6) it follows that

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.7}$$

Therefore, the vector function $\mathbf{V}$ is a solution of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. By virtue of Theorem 7, this BVP has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+ \tag{7.8}$$

and from (6.1), (7.6) and (7.8) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.9}$$

Thus, the homogeneous equation (7.5) admits only the trivial solution, and by virtue of Noether's theorem, Eq. (7.4) is always solvable for an arbitrary vector function $\mathbf{f}$. □

THEOREM 14. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by single-layer potential*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \textit{for } \mathbf{x} \in \Omega^-, \tag{7.10}$$

where $\mathbf{h}$ is a solution of the singular integral equation

$$\mathcal{L}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \textit{for } \mathbf{z} \in S \tag{7.11}$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Proof. We seek a regular solution to the BVP $(II)^-_{\mathbf{0},\mathbf{f}}$ in the form of the single-layer potential (7.10), where $\mathbf{h}$ is the density of the potential and represents the unknown five-component vector function. According to Theorem 9, the vector function $\mathbf{U}$ satisfies Eq. (5.14).

By applying the boundary condition (4.6) and using identity (6.1), from Eq. (7.10) we obtain the singular integral equation (7.11) for the unknown vector function $\mathbf{h}$.

As shown in the proof of Theorem 13, the corresponding homogeneous equation (7.5) associated with Eq. (7.11) admits only the trivial solution. Hence, by Noether's theorem, Eq. (7.11) is always solvable. □

THEOREM 15. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, then a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the double-layer potential*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \textit{for } \mathbf{x} \in \Omega^-, \tag{7.12}$$

where $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{L}^{(3)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \textit{for } \mathbf{z} \in S \tag{7.13}$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Proof. We seek a regular solution to the BVP $(I)^{-}_{\mathbf{0},\mathbf{f}}$ in the form of the double-layer potential (7.12), where $\mathbf{g}$ denotes the potential density and represents the unknown five-component vector function. According to Theorem 10, the vector function $\mathbf{U}$ satisfies Eq. (5.14).

By applying the boundary condition (4.5) and using relations (6.2) and (7.1), substitution into Eq. (7.12) yields the singular integral equation (7.13) for the vector function $\mathbf{g}$.

We now show that Eq. (7.13) is always solvable for an arbitrary vector function $\mathbf{f}$. The corresponding adjoint homogeneous equation associated with Eq. (7.13) can be written as follows

$$\mathcal{L}^{(2)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S, \tag{7.14}$$

where $\mathbf{h}_0$ is the required five-component vector function.

Let $\mathbf{h}_0$ be a solution of Eq. (7.14). By Theorem 9 and Eq. (7.14), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a solution to the BVP $(II)^{+}_{\mathbf{0},\mathbf{0}}$. According to Theorem 7, this BVP admits only the trivial solution, from which we deduce relation (7.8).

Furthermore, by Theorem 9 and relation (7.8), we obtain the boundary condition (7.7), which implies that the vector function $\mathbf{V}(\mathbf{x})$ is a regular solution of the homogeneous BVP $(I)^{-}_{\mathbf{0},\mathbf{0}}$. By Theorem 8, this BVP also admits only the trivial solution, i.e., we have the relation (7.6).

Now, on the basis of Eqs. (7.6), (7.8) and the identity (6.1) we obtain Eq. (7.9). Therefore, Eq. (7.14) has only the trivial solution and by Noether's theorem Eq. (7.13) is always solvable. □

Theorem 16. *If* $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *then a regular solution of problem* $(II)^{+}_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by single-layer potential*

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \textit{for } \mathbf{x} \in \Omega^{+}, \tag{7.15}$$

where $\mathbf{h}$ *is a solution of the singular integral equation*

$$\mathcal{L}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \textit{for } \mathbf{z} \in S, \tag{7.16}$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Proof. We seek a regular solution to the BVP $(II)^{+}_{\mathbf{0},\mathbf{f}}$ in the form of the single-layer potential (7.15), where $\mathbf{h}$ denotes the potential density and represents the unknown five-component vector function. According to Theorem 9, the vector function $\mathbf{U}$ satisfies Eq. (5.1) for $\mathbf{x} \in \Omega^+$. By applying the boundary condition (4.4) and using identity (6.1), substitution into Eq. (7.15) yields the singular integral equation (7.16) for the unknown vector function $\mathbf{h}$.

As established in the proof of Theorem 15, the corresponding homogeneous equation (7.13) associated with Eq. (7.16) admits only the trivial solution. Therefore, equation (7.16) is always solvable. □

8. Conclusion

1. This paper examines the linear theory of MGT thermoviscoelasticity for materials with voids and presents the following main results:

(i) The governing equations for both motion and steady vibrations are proposed.
(ii) The fundamental solution to the system of steady vibration equations is explicitly constructed using four elementary functions, and its essential properties are established.
(iii) Green's first identity is derived and employed as the basis for proving uniqueness theorems for classical solutions of the internal and external BVPs for steady vibrations.
(iv) Surface and volume potentials are introduced, and their key properties are analyzed. Singular integral operators are defined, and their symbolic determinants and indices are computed.
(v) The existence theorems for classical solutions to these BVPs are established using the potential method.

2. Based on the results of this paper, it is possible to:

(i) Introduce the linear model of the MGT thermoviscoelasticity for materials with multiple porosity;
(ii) Investigate the BVPs of steady vibrations of this model using the potential method and the theory of singular integral equations.

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