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An alternative approach to the representation of orthotropic tensor functions in the two-dimensional case

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THE AIM of this paper is to derive in a simple fashion the non-polynomial representations of a class of orthotropic functions in the two-dimensional case. Scalar-valued, vector-valued, symmetric and skew-symmetric tensor-valued functions of the second order have been considered.

1. Introduction

STRUCTURES made of anisotropic materials are often used in engineering practice. Constitutive modelling of the behaviour of such materials has been significantly influenced by the theory of invariants and tensor functions, cf. [6, 18, 24]; vice versa, development of the invariant theory has been stimulated by the constitutive modelling. The reader interested in the fundamentals of the theory of invariants and tensor functions and their applications should refer to [6, 13, 21, 22, 23].

The problem of the determination of the general form of a tensor function of specified order and symmetry depending on tensor arguments consists in finding irreducible sets of scalar invariants and tensor generators; to put it simply, in the determination of the so-called canonical form of the tensor function. Though the theory of tensor function representation has been developed for more than three decades [18, 22, 23], yet no comprehensive, systematic and up-to-date study is available in the relevant literature. The book by SMITH [21] is restricted to the presentation of theoretical results elaborated by this author and his coworkers, by employing classical methods of the group representation theory. SMITH [21] has deliberately focussed on polynomial representations only. Many other complementary contributions exist, however, concerning the general representation of practically important isotropic [3, 14–16, 19, 20, 22–28] and anisotropic [1, 2, 4–6, 10, 12, 21, 29, 30] tensor functions.

Irreducibility of a set of invariants may be understood in two ways:

1. If one determines an integrity basis, then none of its elements can be a *polynomial* in the remaining elements, cf. [22].

2. In the case of a functional or non-polynomial basis, none of its elements can be *a function* of the remaining elements.

Similar characterization pertains to the irreducibility of generators appearing in the canonical form of a tensor function, cf. [3, 6, 16]. To find the polynomial representation of a tensor function it suffices to determine the relevant integrity basis, because the generators are obtained by a simple process of differentiation

[6, 22]. The problem of the non-polynomial representation of a tensor function is more complicated, cf. [3, 19, 20, 25–30]. In the paper by the second author [24], a similar approach was suggested for the determination of generators of non-polynomial tensor functions. This method was next developed by KORSGAARD [14, 15] and used in [11, 12].

In general, the determination of functional bases and generators leads to solving complicated algebraic relations. Hence only some classes of tensor functions are known explicitly. Even when the representations of scalar-, vector- and tensor-valued functions are available, alternative methods of their determination are still proposed, cf. [28, 29].

As is well known, two-dimensional problems are often studied in the continuum mechanics. Thus the problem of the representation of isotropic and anisotropic functions in the two-dimensional case is of interest in itself. However, such two-dimensional representations do not necessarily coincide with those derived directly from the corresponding three-dimensional cases.

The aim of this contribution, precisely formulated in the next section, is to propose an alternative derivation of functional bases and generators for orthotropic functions in the two-dimensional case.

2. Formulation of the problem

The objective of our considerations is the determination of the general form of the following functions:

$$
\begin{aligned}
s &= f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), && i = 1,\dots,I, \quad p = 1,\dots,P, \quad m = 1,\dots,M,\\
\mathbf{t} &= \mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), && \\
\mathbf{S} &= \mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), && \mathbf{S} = \mathbf{S}^t,\\
\mathbf{T} &= \mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m), && \mathbf{T} = -\mathbf{T}^t,
\end{aligned}
\tag{2.1}
$$

in the two-dimensional case. Here $s \in \mathbb{R}$, $\mathbf{t}$, $\mathbf{v}_m \in \mathbb{E}^2$, $\mathbf{S}$, $\mathbf{A}_i \in T^s$ ($\dim T^s = 3$), $\mathbf{T}$, $\mathbf{W}_p \in T^a$ ($\dim T^a = 1$), $T = \mathbb{E}^2 \otimes \mathbb{E}^2 = T^s \oplus T^a$ ($\dim T = 4$), $\mathbb{E}^2$ stands for the two-dimensional Euclidean space and $T^s = \{\mathbf{A} \in T \mid \mathbf{A} = \mathbf{A}^t\}$, $T^a = \{\mathbf{W} \in T \mid \mathbf{W} = -\mathbf{W}^t\}$.

In our 2D case, the orthotropy group S satisfies the condition

$$
\forall\ \mathbf{Q} \in S, \quad \mathbf{Q}\mathbf{M}\mathbf{Q}^t = \mathbf{M},
\tag{2.2}
$$

where $\mathbf{M} = \mathbf{e} \otimes \mathbf{e}$ and the unit vector $\mathbf{e}$ characterises orthotropy, see ([6], p. 51). Obviously we have $\operatorname{tr}\mathbf{M} = \operatorname{tr}\mathbf{M}^2 = 1$.

For each $\mathbf{Q} \in S$, the scalar-valued function f, vector-valued function $\mathbf{f}$, symmetric tensor-valued function $\mathbf{F}$ and skew-symmetric tensor-valued function $\mathbf{G}$

satisfy the conditions:

$$
\begin{aligned}
f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= f(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= \mathbf{f}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\mathbf{Q}^t &= \mathbf{F}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m),\\
\mathbf{Q}\mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\mathbf{Q}^t &= \mathbf{G}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m).
\end{aligned}
\tag{2.3}
$$

By applying I-SHIH LIU theorem [10] (see also [17]) and taking into account (2.2), the invariance requirement (2.3) may be written in the following way:

$$
\begin{aligned}
f(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M}) &= f(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M}) &= \mathbf{f}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^t &= \mathbf{F}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),\\
\mathbf{Q}\mathbf{G}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{M})\mathbf{Q}^t &= \mathbf{G}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^t, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^t, \mathbf{Q}\mathbf{v}_m, \mathbf{Q}\mathbf{M}\mathbf{Q}^t),
\end{aligned}
\tag{2.4}
$$

for each $\mathbf{Q} \in O$, where O denotes the full orthogonal group. Now $\mathbf{M}$ plays the role of a parametric tensor, and the functions f, $\mathbf{f}$, $\mathbf{F}$ and $\mathbf{T}$ depend explicitly on it. We observe that the approach leading to (2.4) has primarily been proposed by BOEHLER [4, 5].

In the sequel we shall derive the functional basis for the scalar function $(2.4)_1$ and generators for the functions $(2.4)_{2-4}$. Our method of determination of the functional basis follows that used by SMITH [19, 20] and KORSGAARD [14, 15] for isotropic functions. Generators will be obtained similarly as in [11, 12, 14, 15], following the idea proposed in the paper by the second author [24].

3. Determination of the orthotropic functional basis

Since the tensor $\mathbf{M}$ appearing in (2.4) is a parametric tensor, the determination of the functional basis is less complicated than in the case of isotropy examined by KORSGAARD [14]. Obviously, in the last case $S = O$, because the invariance with respect to the full orthogonal group has been studied.

To find the functional basis for the orthotropic scalar function $(2.4)_1$, it suffices to consider the following three cases.

CASE 1

In the set of vectors $\{\mathbf{v}_m\}$ $(m = 1, \ldots, M)$ there are vectors non-collinear with the direction of $\mathbf{e}$.

CASE 1.1

At least one vector from the set $\{\mathbf{v}_m\}$, say $\mathbf{v}_1$, is not collinear with $\mathbf{e}$ and $\mathbf{v}_m \neq \mathbf{0}$, $m = 1, \ldots, M$. Then we choose the coordinate system $\{x_\alpha\}$ $(\alpha = 1, 2)$ in such a way that $0x_1$ coincides with $\mathbf{e}$ and $v_1^{(1)} > 0$, $v_2^{(1)} > 0$; here $\mathbf{v}_m = (v_m^{(\alpha)})$. To determine uniquely the representation of the function $(2.4)_1$, it suffices to know the following invariants, since then the components of all arguments are available:

$$
(3.1)\quad
\begin{aligned}
\mathbf{v}_1\cdot\mathbf{M}\mathbf{v}_1 &= v_1^{(1)}v_1^{(1)} \Rightarrow v_1^{(1)} \qquad (v_1^{(1)}>0),\\
\mathbf{v}_1\cdot\mathbf{v}_1 &= v_1^{(1)}v_1^{(1)} + v_2^{(1)}v_2^{(1)} \Rightarrow v_2^{(1)} \qquad (v_2^{(1)}>0),\\
\mathbf{v}_1\cdot\mathbf{M}\mathbf{v}_m &= v_1^{(1)}v_1^{(m)} \Rightarrow v_1^{(m)},\\
\mathbf{v}_1\cdot\mathbf{v}_m &= v_1^{(1)}v_1^{(m)} + v_2^{(1)}v_2^{(m)} \Rightarrow v_2^{(m)},\\
&\left.\begin{aligned}
\mathbf{v}_1\cdot\mathbf{A}_i\mathbf{v}_1 &= A_{11}^{(i)}v_1^{(1)}v_1^{(1)} + 2A_{12}^{(i)}v_1^{(1)}v_2^{(1)} + A_{22}^{(i)}v_2^{(1)}v_2^{(1)},\\
\mathbf{v}_1\cdot\mathbf{A}_i\mathbf{v}_m &= A_{11}^{(i)}v_1^{(1)}v_1^{(m)} + A_{12}^{(i)}\left(v_1^{(1)}v_2^{(m)} + v_1^{(m)}v_2^{(1)}\right) + A_{22}^{(i)}v_2^{(1)}v_2^{(m)},\\
\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m &= A_{11}^{(i)}v_1^{(m)}v_1^{(m)} + 2A_{12}^{(i)}v_1^{(m)}v_2^{(m)} + A_{22}^{(i)}v_2^{(m)}v_2^{(m)},
\end{aligned}\right\} \Rightarrow \mathbf{A}_i\\
\mathbf{v}_1\cdot\mathbf{W}_p\mathbf{v}_m &= W_{12}^{(p)}\left(v_1^{(1)}v_2^{(m)} - v_1^{(m)}v_2^{(1)}\right) \Rightarrow W_{12}^{(p)},
\end{aligned}
$$

provided that $v_1^{(1)}v_2^{(m)} - v_1^{(m)}v_2^{(1)} \neq 0$.

CASE 1.2

Only one vector, say $\mathbf{v} = (v_1, v_2) \in \{\mathbf{v}_m\}$ is not collinear with $\mathbf{e}$, whereas the remaining vectors are zero vectors. We choose the coordinate system similarly as before; then $v_1 > 0$ and $v_2 > 0$. The invariants listed below suffice for the determination of the representation of the function $(2.4)_1$:

$$
(3.2)\quad
\begin{aligned}
\mathbf{v}\cdot\mathbf{M}\mathbf{v} &= v_1v_1 \Rightarrow v_1 \qquad (v_1>0),\\
\mathbf{v}\cdot\mathbf{v} &= v_1^2 + v_2^2 \Rightarrow v_2 \qquad (v_2>0),\\
&\left.\begin{aligned}
\mathbf{v}\cdot\mathbf{A}_i\mathbf{v} &= A_{11}^{(i)}v_1^2 + 2A_{12}^{(i)}v_1v_2 + A_{22}^{(i)}v_2^2,\\
\operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)},\\
\operatorname{tr}\mathbf{MA} &= A_{11}^{(i)},
\end{aligned}\right\} \Rightarrow \mathbf{A}_i\\
\mathbf{v}\cdot\mathbf{M}\mathbf{W}_p\mathbf{v} &= v_1v_2W_{12}^{(p)} \Rightarrow W_{12}^{(p)},
\end{aligned}
$$

where

$$\mathbf{A}_i = \left(A_{\alpha\beta}^{(i)}\right) \qquad (\alpha, \beta = 1, 2).$$

Summarizing, we compile Table 1.

Table 1. Functional basis in Case 1.

$\mathbf{v}_m\cdot\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_n$,	$m = 1, \dots, M$,	$m < n$,
$\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{W}_p\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{W}_p\mathbf{v}_m$,	$i = 1, \dots, I$,	$p = 1, \dots, P$.

CASE 2

We assume that $\mathbf{v}_m = \mathbf{0}$, $m = 1, \dots, M$. Since $\mathbf{M} = \mathbf{e} \otimes \mathbf{e} \neq \mathbf{0}$, hence the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$.

CASE 2.1

Among the tensors $\mathbf{A}_i$ $(i = 1, \dots, I)$ there is none with non-zero off-diagonal components in the coordinate system $\{x_\alpha\}$, such that $0x_1$ and $0x_2$ coincide with the directions of the eigenvectors of $\mathbf{M}$. Let $\mathbf{W} \in \{\mathbf{W}_p\}$. Then the sense of $0x_1$ is chosen in such a way that $W_{12} > 0$. Now one has to know the following invariants:

$$\tag{3.3}\begin{aligned} &\left.\begin{aligned} \operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)}, \\ \operatorname{tr}\mathbf{M}\mathbf{A}_i &= A_{11}^{(i)}, \end{aligned}\right\} \Rightarrow A_{11}^{(i)} \text{ and } A_{22}^{(i)}, \\ &\operatorname{tr}\mathbf{W}^2 = -2W_{12}^2 \Rightarrow W_{12} \qquad (W_{12} > 0), \\ &\operatorname{tr}\mathbf{W}\mathbf{W}_p = -2W_{12}W_{12}^{(p)} \Rightarrow W_{12}^{(p)}. \end{aligned}$$

CASE 2.2

Let $\mathbf{B} \in \{\mathbf{A}_i\}$ denote a tensor with non-zero off-diagonal components. The positive direction of $0x_1$ is chosen in such a way that $B_{12} > 0$. The set of invariants is:

$$\tag{3.4}\begin{aligned} &\left.\begin{aligned} \operatorname{tr}\mathbf{A}_i &= A_{11}^{(i)} + A_{22}^{(i)}, \\ \operatorname{tr}\mathbf{M}\mathbf{A}_i &= A_{11}^{(i)}, \end{aligned}\right\} \Rightarrow A_{11}^{(i)} \text{ and } A_{22}^{(i)}, \\ &\left.\begin{aligned} \operatorname{tr}\mathbf{B} &= B_{11} + B_{22}, \\ \operatorname{tr}\mathbf{M}\mathbf{B} &= B_{11}, \end{aligned}\right\} \Rightarrow B_{11} \text{ and } B_{22}, \\ &\operatorname{tr}\mathbf{B}^2 = B_{11}^2 + 2B_{12}^2 + B_{22}^2, \Rightarrow B_{12}, \\ &\operatorname{tr}\mathbf{B}\mathbf{A} = B_{11}A_{11}^{(i)} + B_{22}A_{22}^{(i)} + 2B_{12}A_{12}^{(i)}, \Rightarrow A_{12}^{(i)}, \\ &\operatorname{tr}\mathbf{M}\mathbf{B}\mathbf{W}_p = -B_{12}W_{12}^{(p)}, \Rightarrow W_{12}^{(p)}. \end{aligned}$$

By applying formulas (3.3) and (3.4) we construct Table 2.

Table 2. Functional basis in Case 2.

$\operatorname{tr}\mathbf{A}_i$, $\operatorname{tr}\mathbf{A}_i^2$, $\operatorname{tr}\mathbf{M}\mathbf{A}_i$, $\operatorname{tr}\mathbf{A}_i\mathbf{A}_j$,	$i, j = 1, \dots, I$,	$i < j$,
$\operatorname{tr}\mathbf{W}_p^2$, $\operatorname{tr}\mathbf{W}_p\mathbf{W}_q$, $\operatorname{tr}\mathbf{M}\mathbf{A}_i\mathbf{W}_p$,	$p, q = 1, \dots, P$,	$p < q$.

CASE 3

All vectors $\mathbf{v}_m$ $(m = 1, \dots, M)$ have the form $\mathbf{v}_m = c_m\mathbf{e}$. Let $\mathbf{v} \in \{\mathbf{v}_m\}$, $\mathbf{v} = c\mathbf{e}$, and choose the coordinate system in such way that $c > 0$. Then we have

$$\tag{3.5} \mathbf{v}\cdot\mathbf{v} = c^2, \Rightarrow c \quad (c > 0), \qquad \mathbf{v}\cdot\mathbf{M}\mathbf{v}_m = cc_m \Rightarrow c_m .$$

The remaining invariants are derived similarly as in Case 2.

Summarizing all three cases: 1, 2 and 3, we obtain the orthotropic functional basis for the two-dimensional problem.

The last table coincides with ZHENG's results [29], who has however used a different method.

BOEHLER [4, 5, 6] determined functional bases provided that functions appearing in (1) depend only on symmetric tensors $\mathbf{A}_i$. In the two-dimensional case, Boehler's results correspond to the first row of our Table 3. This author approached the two-dimensional case through the three-dimensional one by using Cayley-Hamilton theorem, cf. also [21]. The method of determination of a functional basis employed in [4, 5, 6] and based on Cayley-Hamilton theorem, proves that the functional basis is also the integrity basis, see also the first row of Table 3.

Table 3. Functional basis for the orthotropic scalar-valued function $(2.4)_1$.

$\operatorname{tr}\mathbf{A}_i$, $\operatorname{tr}\mathbf{A}_i^2$, $\operatorname{tr}\mathbf{MA}_i$, $\operatorname{tr}\mathbf{A}_i\mathbf{A}_j$,	$i,j = 1,\ldots,I,$	$i<j,$
$\operatorname{tr}\mathbf{W}_p^2$, $\operatorname{tr}\mathbf{W}_p\mathbf{W}_q$, $\operatorname{tr}\mathbf{MA}_i\mathbf{W}_p$,	$p,q = 1,\ldots,P,$	$p<q,$
$\mathbf{v}_m\cdot\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{M}\mathbf{v}_n$,	$m,n = 1,\ldots,M,$	$m<n,$
$\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m$, $\mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{W}_p\mathbf{v}_n$, $\mathbf{v}_m\cdot\mathbf{MW}_p\mathbf{v}_m$.		

ADKINS [1, 2] determined integrity basis, in the two- and three-dimensional cases, for arbitrary second order tensors, under the condition of linearity of invariants with respect to each argument. Consequently, two-dimensional reduction of the invariants in the case of transverse isotropy characterized by the parametric tensor $\mathbf{M}$ does not yield the invariants listed in the first and second row of Table 3. It is worth noting that the tensor $\mathbf{M}$ describes only one of the five possible cases of 3D transverse isotropy, cf. [29].

Let $\mathbf{D}_i \in T$ $(i = 1,\ldots,I)$ be arbitrary two-dimensional second order tensors, not necessarily symmetric. Assuming that one of the axis of the Cartesian coordinate system coincides with $\mathbf{e}$, Adkins' integrity basis is given by

$$
\begin{aligned}
&D^{(i)}_{11}, \quad D^{(i)}_{\alpha\beta},\\
&D^{(i)}_{\alpha\beta}D^{(j)}_{\beta\alpha}, \quad D^{(i)}_{1\alpha}D^{(j)}_{\alpha 1}, \qquad \alpha,\beta,\gamma = 1,\ 2,\\
&D^{(i)}_{1\alpha}D^{(k)}_{\alpha\beta}D^{(j)}_{\beta 1}, \qquad i,j,k,l = 1,\ldots,I,\\
&D^{(i)}_{1\alpha}D^{(k)}_{\alpha\beta}D^{(l)}_{\beta\gamma}D^{(j)}_{\gamma 1}, \qquad i>j>k>l,
\end{aligned}
\tag{3.6}
$$

where

$$\mathbf{D}_i = D^{(i)}_{\alpha\beta}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta \qquad \text{and} \qquad \mathbf{e}_1 = \mathbf{e}.$$

4. Determination of generators of an orthotropic vector-valued function

In this section we shall derive the general form of the vector-valued function $(2.4)_2$. To this end we consider the scalar function, cf. [11, 14, 24]

$$g = \mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m)\cdot\mathbf{d} = f_\alpha d_\alpha , \tag{4.1}$$

linear in $\mathbf{d}$. Thus we may write

$$g(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{d}) = \widehat{g}(I_r, J_s) = \sum_{s=1}^{S} \psi_s(I_r) J_s , \tag{4.2}$$

where I_r are the invariants listed in Table 3, while J_s are the following invariants, linear in $\mathbf{d}$:

$$\mathbf{d}\cdot\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{M}\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{A}_i\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{W}_p\mathbf{v}_m . \tag{4.3}$$

They are obtained by using the procedure outlined in the previous section. In fact, since in (4.1) a vector $\mathbf{d}$ appears, therefore we do not consider Case 2. In Case 1 the invariants $\mathbf{d}\cdot\mathbf{v}_m$, $\mathbf{d}\cdot\mathbf{M}\mathbf{v}_m$, permit us to determine $\mathbf{d}$ uniquely. Considering Case 3, since

$$\mathbf{d}\cdot\mathbf{v}_m = d_1 c_m \Rightarrow d_1 ,$$

we must additionally examine the following two situations.

CASE 3.1

At least one of the tensors, say $\mathbf{A} \in \{\mathbf{A}_i\}$, is not singular, that is it has two different eigenvalues. Then the two invariants: $\mathbf{d}\cdot\mathbf{v}_m$, $\mathbf{d}\cdot\mathbf{A}\mathbf{v}_m$ determine the components d_α $(\alpha = 1, 2)$ of $\mathbf{d}$ uniquely.

CASE 3.2

At least one of the tensors, say $\mathbf{W} \in \{\mathbf{W}_p\}$, is such that the corresponding axial vector [22] is not collinear with $\mathbf{e}$. Then

$$\mathbf{d}\cdot\mathbf{v}_m , \quad \mathbf{d}\cdot\mathbf{W}\mathbf{v}_m \Rightarrow d_1 \text{ and } d_2 ,$$

and $\mathbf{d}$ is determined uniquely.

We observe that if in Case 3 the situations covered by Cases 3. 1 and 3. 2 do not occur, then it suffices to know the invariant $\mathbf{d}\cdot\mathbf{v}_m = d_1 c_m$, because the vector-valued function has the form $\mathbf{f} = \phi\mathbf{e}$, where ϕ stands for an invariant.

The canonical form of the vector-valued function $(2.4)_2$ is given by

$$\mathbf{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) = \frac{\partial \widehat{g}}{\partial \mathbf{d}} = \sum_{s=1}^{S} \psi_s(I_r) \frac{\partial J_s}{\partial \mathbf{d}} = \sum_{s=1}^{S} \psi_s(I_r) \mathbf{g}_s . \tag{4.4}$$

The generators $\mathbf{g}_s$ are listed in Table 4 and coincide with the results due to ZHENG [29].

Table 4. Generators of the orthotropic vector-valued function $(2.4)_2$.

$\mathbf{v}_m$,	$\mathbf{M}\mathbf{v}_m$,	$\mathbf{A}_i\mathbf{v}_m$,	$\mathbf{W}_p\mathbf{v}_m$,	$m = 1, \ldots, M$,	$i = 1, \ldots, I$,	$p = 1, \ldots, P$.

5. Determination of generators of the orthotropic symmetric tensor-valued function

Proceeding similarly as in the previous section we take

$$h = \operatorname{tr} \mathbf{F}\mathbf{C}, \tag{5.1}$$

where $\mathbf{C}$ is a symmetric second-order tensor. The scalar-valued function h has now the form

$$h(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{C}) = \widehat{h}(I_r, J_s) = \sum_{s=1}^{S} \phi_s(I_r) J_s\,, \tag{5.2}$$

where I_r are the invariants listed in Table 3, and J_s are linear in $\mathbf{C}$:

$$\operatorname{tr}\mathbf{C}, \quad \operatorname{tr}\mathbf{M}\mathbf{C}, \quad \operatorname{tr}\mathbf{C}\mathbf{A}_i\,, \quad \operatorname{tr}\mathbf{C}\mathbf{M}\mathbf{W}_p\,, \quad \mathbf{v}_m\cdot\mathbf{C}\mathbf{v}_m\,, \quad \mathbf{v}_m\cdot\mathbf{C}\mathbf{v}_n\,. \tag{5.3}$$

To justify (5.3) one has to consider the following three cases.

CASE 1.1

Let $\mathbf{v}_1$, $\mathbf{v}_2 \in \{\mathbf{v}_m\}$ be such that $\det[v_\alpha^{(1)} v_\beta^{(2)}] \neq 0$. Then by using the invariants $\mathbf{v}_1\cdot\mathbf{C}\mathbf{v}_1$, $\mathbf{v}_2\cdot\mathbf{C}\mathbf{v}_2$ and $\mathbf{v}_1\cdot\mathbf{C}\mathbf{v}_2$ we determine $\mathbf{C}$ uniquely. In Case 1.2 one can also calculate these invariants, because $\mathbf{v}_1$ and $\mathbf{e}$ are not collinear.

CASE 2.1

Knowing the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{M}\mathbf{C}$, $\operatorname{tr}\mathbf{C}\mathbf{M}\mathbf{W}$ one determines $\mathbf{C}$ uniquely.

If in Case 2.1 all skew-symmetric tensors disappear or their axial vectors are collinear with $\mathbf{e}$, then it suffices to know the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{M}\mathbf{C}$, because $\mathbf{F}$ has diagonal form.

CASE 2.2

Since the off-diagonal components of the tensor $\mathbf{B}$ are non-zero, it suffices to know the invariants: $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}\mathbf{M}\mathbf{C}$ and $\operatorname{tr}\mathbf{C}\mathbf{B}$.

All in all, to satisfy the cases considered, the set of invariants linear in $\mathbf{C}$ has to be specified by (5.3).

The canonical form of the tensor-valued function $(2.4)_3$ is given by

$$\mathbf{F}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) = \frac{1}{2}\left(\frac{\partial h}{\partial \mathbf{C}} + \frac{\partial h}{\partial \mathbf{C}^T}\right) = \frac{\partial h}{\partial \mathbf{C}} = \sum_{s=1}^{S} \phi_s(I_r)\frac{\partial J_s}{\partial \mathbf{C}} = \sum_{s=1}^{S} \overline{\phi}_s(I_r)\mathbf{F}_s\,. \tag{5.4}$$

The results are summarized in Table 5. The generators $\mathbf{F}_s$ are the same as those obtained by ZHENG [29]. The case considered by BOEHLER [4, 6] is covered by the first row of Table 5.

Table 5. Generators of the orthotropic, symmetric tensor-valued function $(2.4)_3$.

$\mathbf{I}$, $\mathbf{M}$, $\mathbf{A}_i$,	$i = 1, \ldots, I$,	
$\mathbf{M}\mathbf{W}_p - \mathbf{W}_p\mathbf{M}$,	$p = 1, \ldots, P$,	
$\mathbf{v}_m \otimes \mathbf{v}_m$, $\mathbf{v}_m \otimes \mathbf{v}_n + \mathbf{v}_n \otimes \mathbf{v}_m$,	$m, n = 1, \ldots, M$,	$m < n$.

6. Determination of generators of the orthotropic skew-symmetric tensor-valued function

We begin by constructing the scalar function [14, 24]

$$k = \operatorname{tr} \mathbf{T}\mathbf{X}, \tag{6.1}$$

where $\mathbf{X}$ is a skew-symmetric tensor. Hence we may write

$$k(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{X}) = \widehat{k}(I_r, K_s) = \sum_{s=1}^{S} \phi_s(I_r) K_s\,, \tag{6.2}$$

where K_s are the invariants, linear in $\mathbf{X}$:

$$\operatorname{tr} \mathbf{M}\mathbf{A}_i\mathbf{X}, \quad \operatorname{tr} \mathbf{X}\mathbf{W}_p\,, \quad \mathbf{v}_m \cdot \mathbf{M}\mathbf{X}\mathbf{v}_m\,, \quad \mathbf{v}_m \cdot \mathbf{X}\mathbf{v}_n\,. \tag{6.3}$$

To justify (6.3) we have to examine the following cases.

CASE 1.1

$$\mathbf{v}_m \cdot \mathbf{X}\mathbf{v}_n = X_{12} \left(v_1^{(m)} v_2^{(n)} - v_1^{(n)} v_2^{(m)} \right), \Rightarrow X_{12}\,.$$

CASE 1.2

$$\mathbf{v}_m \cdot \mathbf{M}\mathbf{X}\mathbf{v}_n = X_{12} v_1^{(m)} v_2^{(m)}, \Rightarrow X_{12}\,.$$

CASE 2.1

$$\operatorname{tr} \mathbf{X}\mathbf{W}_p \Rightarrow X_{12}^{(p)}\,.$$

CASE 2.2

$$\operatorname{tr} \mathbf{M}\mathbf{B}\mathbf{X} = -B_{12} X_{12}\,, \qquad B_{12} > 0, \Rightarrow X_{12}\,.$$

Case 3 is treated similarly as Cases 2.1 and 2.2.

The canonical form of the function $\mathbf{T}$ is given by

$$\begin{aligned} \mathbf{T}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m) &= \frac{1}{2} \left(\frac{\partial k}{\partial \mathbf{X}} - \frac{\partial k}{\partial \mathbf{X}^T} \right) \\ &= \frac{1}{2} \sum_{s=1}^{S} \phi_s(I_r) \left(\frac{\partial K_s}{\partial \mathbf{X}} - \frac{\partial K_s}{\partial \mathbf{X}^T} \right) = \sum_{s=1}^{S} \widetilde{\phi}_s(I_r) \mathbf{T}_s\,. \end{aligned} \tag{6.4}$$

The generators of $\mathbf{T}_s$ are listed in Table 6. They coincide with those obtained by ZHENG [29].

Table 6. Generators of the orthotropic, skew-symmetric tensor-valued function $(2.4)_4$.

$\mathbf{M}\mathbf{A}_i - \mathbf{A}_i\mathbf{M}$, $\mathbf{W}_p$,	$i = 1, \ldots, I$,	$p = 1, \ldots, P$,
$\mathbf{v}_m \otimes \mathbf{M}\mathbf{v}_m - \mathbf{M}\mathbf{v}_m \otimes \mathbf{v}_m$, $\mathbf{v}_m \otimes \mathbf{v}_n - \mathbf{v}_n \otimes \mathbf{v}_m$,	$m, n = 1, \ldots, M$,	$m < n$.

7. Equivalent functional bases and sets of generators

ZHENG [30] determined an alternative form of the functional basis and generators in comparison with the results of his first paper [29]. In [30] the representations of functions (2.1) corresponding to all anisotropy groups have been investigated. Then orthotropy group is the group C_{2v} (cf. also [21]) and the parametric tensor $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 . \tag{7.1}$$

Here $\mathbf{e}_\alpha$ ($\alpha = 1, 2$) are unit vectors specifying the directions of orthotropy. By setting $\mathbf{e}_1 = \mathbf{e}$, we readily obtain

$$\mathbf{K} = 2\mathbf{M} - \mathbf{I}. \tag{7.2}$$

This relation enables the passage from our results to those due to ZHENG [30] in the two-dimensional case of orthotropy.

The results obtained by ZHENG [29, 30] and in this contribution can be applied to the determination of representations of the following functions:

$$\begin{aligned}
\widetilde{s} &= \widetilde{f}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \quad i = 1, \ldots, I, \quad p = 1, \ldots, P, \quad m = 1, \ldots, M,\\
\widetilde{\mathbf{t}} &= \widetilde{\mathbf{f}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}),\\
\widetilde{\mathbf{S}} &= \widetilde{\mathbf{F}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \qquad \widetilde{\mathbf{S}} = \widetilde{\mathbf{S}}^t,\\
\widetilde{\mathbf{T}} &= \widetilde{\mathbf{G}}(\mathbf{A}_i, \mathbf{W}_p, \mathbf{v}_m, \mathbf{H}), \qquad \widetilde{\mathbf{T}} = -\widetilde{\mathbf{T}}^t,
\end{aligned} \tag{7.3}$$

where $\mathbf{H}$ is a symmetric, positive definite tensor. Its eigenvalues are denoted by H_1 and H_2, $H_1 > H_2$. Now we have

$$\begin{aligned}
\mathbf{H} &= H_1\mathbf{e}_1 \otimes \mathbf{e}_1 + H_2\mathbf{e}_2 \otimes \mathbf{e}_2 ,\\
\mathbf{H} &= H_1\mathbf{M} + H_2(\mathbf{I} - \mathbf{M}).
\end{aligned} \tag{7.4}$$

Consequently one can easily determine the representations of the functions appearing in (7.3).

The last case is important for applications if $\mathbf{H}$ plays the role of a fabric tensor, cf. [7, 8, 9]. This tensor is sometimes used to model the mechanical behaviour of materials as different as soils [6] and bones [7–9].

In the case when $H_1 = H_2$, $\mathbf{H}$ is a spherical tensor and the representations of functions (7.3) coincide with those derived by KORSGAARD [14]; then the tensor $\mathbf{H}$ does not appear in these functions.

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References

1. J.E. ADKINS, *Symmetry relations for orthotropic and transversely isotropic materials*, Arch. Rat. Mech. Anal., **4**, 193–213, 1960.
2. J.E. ADKINS, *Further symmetry relations for transversaly isotropic materials*, Arch. Rat. Mech. Anal., **5**, 263–274, 1960.
3. J.P. BOEHLER, *On irreducible representations for isotropic scalar functions*, ZAMM, **57**, 323–327, 1977.
4. J.P. BOEHLER, *Lois de comportment anisotrope des milieux continus*, J. Méc., **17**, 153–190, 1978.
5. J.P. BOEHLER, *A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy*, ZAMM, **59**, 157–167, 1979.
6. J.P. BOEHLER [Ed.], *Applications of tensor functions in solid mechanics*, CISM Courses and Lectures No. 292, Springer-Verlag, Wien-New York 1987.
7. S.C. COWIN, *The relationship between the elasticity tensor and the fabric tensor*, Mech. Mat., **4**, 137–147, 1985.
8. S.C. COWIN, *Fabric dependence of an anisotropic strength criterion*, Mech. Mat., **5**, 251–260, 1986.
9. S.C. COWIN, *Wolff's law of trabecular architecture at remodeling equilibrium*, J. Biomechanical Engng., **108**, 83–88, 1986.
10. I-SHIH LIU, *On representations of anisotropic invariants*, Int. J. Engng. Sci., **20**, 1099–1109, 1982.
11. S. JEMIOŁO, *Some comments on the representation of vector-valued isotropic function*, J. Theoret. and Appl. Mech., **31**, 121–125, 1993.
12. S. JEMIOŁO and M. KWIECIŃSKI, *On irreducible number of invariants and generators in the constitutive relationships*, Engng. Trans., **39**, 241–253, 1990.
13. E. KIRAL and A.C. ERINGEN, *Constitutive equations of nonlinear electromagnetic-elastic crystals*, Springer-Verlag, New York 1990.
14. J. KORSGAARD, *On the representation of two-dimensional isotropic functions*, Int. J. Engng. Sci., **28**, 653–662, 1990.
15. J. KORSGAARD, *On the representation of symmetric tensor-valued isotropic functions*, Int. J. Engng. Sci., **28**, 1331–1346, 1990.
16. S. PENNISI and M. TROVATO, *On the irreducibility of Professor G.F. Smith's representations for isotropic functions*, Int. J. Engng. Sci., **25**, 1059–1065, 1987.
17. J. RYCHLEWSKI, *Symmetry of causes and effects* [in Polish], PWN, Warszawa 1991.
18. J. RYCHLEWSKI and J.M. ZHANG, *On representation of tensor functions: A review*, Advances in Mech., **14**, 75–94, 1991.
19. G.F. SMITH, *On a fundamental error in two papers of C.C. Wang*, Arch. Rat. Mech. Anal., **36**, 161–165, 1970.
20. G.F. SMITH, *On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors*, Int. J. Engng. Sci., **9**, 899–916, 1971.
21. G.F. SMITH, *Constitutive equations for anisotropic and isotropic materials*, North-Holland, Amsterdam, London, New York, Toronto 1994.
22. A.J.M. SPENCER, *Theory of invariants*, [in:] Continuum Physics, Vol. I, A.C. ERINGEN [Ed.], Academic Press, 1971.
23. J.J. TELEGA, *Theory of invariants: from Boole to the present. Tensor functions and concomitants* [in Polish], [in:] Methods of Functional Analysis in Plasticity, J.J. TELEGA [Ed.], pp. 331–361, Ossolineum, Wrocław 1981.
24. J.J. TELEGA, *Some aspects of invariant theory in plasticity, Part I. New results relative to representation of isotropic and anisotropic tensor functions*, Arch. Mech., **36**, 147–162, 1984.

25. C.C. WANG, *On representations for isotropic functions. Part I and II*, Arch. Rat. Mech. Anal., **33**, 249–287, 1969.

26. C.C. WANG, *A new representation theorem for isotropic functions. Part I and II*, Arch. Rat. Mech. Anal., **36**, 166–223, 1970.

27. C.C. WANG, *Corrigendum*, Arch. Rat. Mech. Anal., **43**, 392–395, 1971.

28. Q.-S. ZHENG, *On the representations for isotropic vector-valued, symmetric tensor-valued and skew-symmetric tensor-valued functions*, Int. J. Engng. Sci., **31**, 1013–1024, 1993.

29. Q.-S. ZHENG, *On transversely isotropic, orthotropic and relative isotropic functions of symmetric tensors and vectors. Part I-V*, Int. J. Engng. Sci., **31**, 1399–1453, 1993.

30. Q.-S. ZHENG, *Two-dimensional tensor function representation for all kinds of material symmetry*, Proc. R. Soc. Lond., **A443**, 127–138, 1993.

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