Random elastic media:
Why zero mean stress does not imply zero mean strain

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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IN THIS NOTE we are concerned with (linearized) elastic media that are heterogeneous on the microscale and homogeneous on the macroscale. We assume the validity of an ergodic hypothesis so that we can form ensemble averages such as mean stress $\langle \sigma \rangle$ and mean strain $\langle \varepsilon \rangle$. We argue that $\langle \sigma \rangle = 0$ does not, in general, imply that also $\langle \varepsilon \rangle = 0$ (end of Sec. 2). This is the case, for instance, when the distribution of the stress sources (external forces or incompatibilities) are correlated with the spatial distribution of the local elastic moduli. It is shown how problems of this type can be treated.

1. Introduction

IN THIS WORK we are concerned with *linear elastic* media that are heterogeneous on the microscale and, at the same time, homogeneous on the macroscale. Many composite materials and also polycrystals can be, at least approximately, in such a state. Under certain conditions such a medium can be treated as a so-called *effective medium* whose material properties are described by a constant effective tensor C^{eff} of elastic moduli. The main interest here lies in the situation where such a description is *not* possible. Among others, we pay attention to the following problem: If the mean stress in the medium is zero, does it follow that also the mean strain vanishes? The answer is "No". I have treated this question previously [1, 2], but perhaps not convincingly enough, because I continue to receive remarks of this type: "Many people have claimed that since the average stress vanishes, the average strain also vanishes." In the following we show that this is not generally true. To make our investigations lucid we use a symbolic notation which is largely free of tensor indices. Various quantities are integral or differential operators with tensorial kernels. Integral operators are underlined by double line, such as $\underline{\underline{\Gamma}}$ or $\underline{\underline{\gamma}}$, differential operators are underlined by a single line, as $\underline{L}$. Instead of specifying positions in the medium by $r_1, r_2, r_3 \ldots$ or $r', r'', r''' \ldots$ we shall indicate them just by numbers, i.e. (1), (2), (3) ... Explanations are often somewhat short. For a more detailed representation the reader is referred to the review [3], even though this note goes beyond this review.

2. Averages of stress and strain

Assume that the medium characterized in Sec.1 be loaded by volume forces $f_k(1)$ and surface forces $t_k(1)$. Applying the method of Green's functions, the displacement (vector) field $u_i(1)$ can be represented as

$$(2.1) \qquad u_i(1) = \int_V \gamma_{ik}(1,2)\, f_k(2)\, dV_2 + \int_S \gamma_{ik}(1,2)\, t_k(2)\, dS_2,$$

where the integrations extend over the volume V and the surface S of the medium. For convenience (1) is abbreviated in symbolic notation as

$$(2.2) \qquad u = \underline{\underline{\gamma}} F,$$

where $\underline{\underline{\gamma}}$ is the *displacement Green's operator* and F comprises the volume and surface forces. (2.1), or (2.2), is the general solution of the traction boundary value problem of an arbitrary, in general heterogeneous or microheterogeneous medium. u, γ and F are, in our investigation, random functions of position.

Since we are interested in macroscopic (ensemble) averages (symbol $\langle \cdot \rangle$) rather than in the local quantities, we take the ensemble average of (2.2):

$$(2.3) \qquad \begin{aligned} \langle u \rangle &= \langle \underline{\underline{\gamma}} F \rangle = \langle \underline{\underline{\gamma}} \rangle \langle F \rangle + \langle \underline{\underline{\gamma}}' F' \rangle, \\ F' &\equiv F - \langle F \rangle, \qquad \underline{\underline{\gamma}}' \equiv \underline{\underline{\gamma}} - \langle \underline{\underline{\gamma}} \rangle, \qquad \langle F' \rangle = 0, \qquad \langle \underline{\underline{\gamma}}' \rangle = 0. \end{aligned}$$

Working with ensemble averages (rather than with volume averages) is convenient because ensemble averaging commutes with differentiating and integrating. This however requires the validity of an ergodic hypothesis, which essentially means that the situation is "sufficiently statistical". The statistics of $\underline{\underline{\gamma}}$ is determined by that of c, the (4-th rank) tensor of the microheterogeneous moduli.

From (2.3) it follows that

$$(2.4) \qquad \langle u \rangle = \langle \underline{\underline{\gamma}} \rangle \langle F \rangle,$$

provided $\langle \underline{\underline{\gamma}}' F' \rangle$ vanishes, i.e. if, according to a well-known theorem of probability theory, the distribution of the forces is not correlated with that of the (microheterogeneous) elastic moduli. This "no-correlation" is the case in many applications (see below). In this note we shall admit correlations and give realizations below.

The strain tensor derived from (2.1) and from the equilibrium conditions $f = \nabla\sigma$, $t = n\sigma$ (n being the external unit vector normal to the surface) by a routine calculation is

$$(2.5) \qquad \varepsilon_{ij}(1) = \int_V \Gamma_{ijkl}(1,2)\, \sigma_{kl}(2)\, dV_2, \qquad \Gamma_{ijkl}(1,2) \equiv \partial_j\, \partial_{l'}\, \gamma_{ik}(1,2)\, |_{(ij)(kl)}$$

which is abbreviated as

$$(2.6) \qquad \varepsilon = \underline{\underline{\Gamma}} \, \sigma.$$

Here $\underline{\underline{\Gamma}}$ is the so-called *modified* or *strain Green's function*, a 2-point, 4-th rank tensor field, and (ij), (kl) denotes symmetrization in the subscripts concerned. Note that the abbreviation introduced in (2.2) involves vector and 2-nd rank tensor fields, whereas that of (2.6) concerns 2-nd and 4-th rank tensor fields.

Taking the average of (2.6) gives us the mean strain tensor

$$(2.7) \qquad \langle \varepsilon \rangle = \langle \underline{\underline{\Gamma}} \, \sigma \rangle = \langle \underline{\underline{\Gamma}} \rangle \langle \sigma \rangle + \langle \underline{\underline{\Gamma}}' \, \sigma' \rangle,$$

where the primed quantities are again deviations from the mean. In (2.7) $\langle \underline{\underline{\Gamma}} \rangle \langle \sigma \rangle$ derives from $\langle \underline{\gamma} \rangle \langle F \rangle$ and $\langle \underline{\underline{\Gamma}}' \sigma' \rangle$ vanishes when $\langle \underline{\gamma}' F' \rangle$ does. In analogy to (2.4) we have

$$(2.8) \qquad \langle \varepsilon \rangle = \langle \underline{\underline{\Gamma}} \rangle \langle \sigma \rangle,$$

provided that u and F, thus $\underline{\underline{\Gamma}}$ and σ are mutually uncorrelated. By comparison with (2.6) $\langle \underline{\underline{\Gamma}} \rangle$ is then the modified Green's function of a random elastic medium with an effective compliance tensor defined by

$$(2.9) \qquad \langle \varepsilon \rangle = S^{\mathrm{eff}} \langle \sigma \rangle, \qquad S^{\mathrm{eff}} = (C^{\mathrm{eff}})^{-1}.$$

Note that contrary to $\langle \underline{\underline{\Gamma}} \rangle$, S^{eff} is not an operator itself, but derives from the operator $\langle \underline{\underline{\Gamma}} \rangle$ according to

$$(2.10) \qquad S^{\mathrm{eff}}_{ijkl} = \int_V \langle \Gamma_{ijkl}(1,2) \rangle \, dV_2 \, ,$$

cf. [3], p. 265.

In the "no-correlation" case $\langle \sigma \rangle = 0$ leads to $\langle \varepsilon \rangle = 0$ and *vice versa*. Obviously this is not so in the more general case of (2.7), when $\langle \underline{\underline{\Gamma}}' \sigma' \rangle \neq 0$.

3. The general scheme

In the frame of linearized elasticity theory the equation

$$(3.1) \qquad u = \underline{\gamma} \, F$$

with the meaning of F as defined in Sec. 2 solves the deterministic problem of finding the displacement field from the knowledge of the external force distribution in the body. Our interest lies in situations with random distributions. That means that we have to extract the random information possibly contained in (3.1). This is usually done with reference to a homogeneous isotropic, but otherwise quite arbitrary *comparison medium* that has the same form and load as the real medium of interest. The named properties of the comparison medium allow us to calculate its Green's function, which therefore will be treated as known.

Let
(3.2) $$\underline{L}\,u = F, \qquad \underline{L}^\circ\,u^\circ = F$$

be the (statical) field equations (in volume and on surface) of the real and the comparison medium, respectively. The displacement Green's operators are $\underline{\gamma}$ and $\underline{\gamma}^\circ$; they obey the conditions

(3.3) $$\underline{L}\,\underline{\gamma} = I, \qquad \underline{L}^\circ\,\underline{\gamma}^\circ = I$$

(the Green's operators $\underline{\gamma}$ are inverse to the field operators $\underline{L}$. This property *defines* the Green's functions.) In (3.3), set $\underline{L} = \underline{L}^\circ + \delta\underline{L}$ and multiply from the left by $\underline{\gamma}^\circ$ to obtain, after solution for $\underline{\gamma}$,

(3.4) $$\begin{aligned}\underline{\gamma} &= (I + \underline{\gamma}^\circ\,\delta\underline{L})^{-1}\underline{\gamma}^\circ \\ &= (I - \underline{\gamma}^\circ\,\delta\underline{L} + \underline{\gamma}^\circ\,\delta\underline{L}\,\underline{\gamma}^\circ\,\delta\underline{L} - \underline{\gamma}^\circ\,\delta\underline{L}\,\underline{\gamma}^\circ\,\delta\underline{L}\,\underline{\gamma}^\circ\,\delta\underline{L} + \cdots - \cdots)\underline{\gamma}^\circ.\end{aligned}$$

Here
(3.5) $$\delta\underline{L} = \nabla\delta c\nabla, \qquad \delta c \equiv c - C^\circ,$$

c denoting the tensor of local elastic moduli, C° – that of the comparison medium. δc and $\delta\underline{L}$ are random functions of position. All the quantities appearing in (3.4) have vector or tensor character. We have used a symbolic notation in which indices and their positions do not occur. They can easily be determined a *posteriori*. If so, then the positions of the $\underline{\gamma}^\circ$ and $\delta\underline{L}$ in (3.4) are not important. In actual calculations one uses indices, their positions follow easily from (3.4). Thus (3.4) may also be written as

(3.6) $$\underline{\gamma} = (I - \underline{\gamma}^\circ\,\delta\underline{L} + \underline{\gamma}^\circ\,\underline{\gamma}^\circ\,\delta\underline{L}\,\delta\underline{L} - \underline{\gamma}^\circ\,\underline{\gamma}^\circ\,\underline{\gamma}^\circ\,\delta\underline{L}\,\delta\underline{L}\,\delta\underline{L} + \cdots - \cdots)\underline{\gamma}^\circ.$$

This equation relates the Green's function $\underline{\gamma}$ of the, in general, complicated real medium to the Green's function $\underline{\gamma}^\circ$ of the simpler comparison medium. It is an implicit equation, because $\delta\underline{L}\;(\equiv \underline{L} - \underline{L}^\circ)$ contains $\underline{\gamma}$ which is the inverse of $\underline{L}$.

If the elastic moduli (the components of c) of the real medium are randomly distributed, then the averaged Green's function $\langle\underline{\gamma}\rangle$ follows from (3.6), since $\underline{\gamma}^\circ$ is nonrandom, as

(3.7) $$\langle\underline{\gamma}\rangle = (I - \underline{\gamma}^\circ\langle\delta L\rangle + \underline{\gamma}^\circ\,\underline{\gamma}^\circ\langle\delta\underline{L}\,\delta\underline{L}\rangle - \underline{\gamma}^\circ\,\underline{\gamma}^\circ\,\underline{\gamma}^\circ\langle\delta\underline{L}\,\delta\underline{L}\,\delta\underline{L}\rangle + \cdots - \cdots)\underline{\gamma}^\circ.$$

To calculate $\langle\underline{\gamma}\rangle$ from this equation we need the statistical information in the form of n-point correlation functions of the elastic operator $\underline{L}$, namely $\langle\delta\underline{L}\rangle$, $\langle\delta\underline{L}\,\delta\underline{L}\rangle$, $\langle\delta\underline{L}\,\delta\underline{L}\,\delta\underline{L}\rangle$, i.e. $\langle\delta\underline{L}^n\rangle$, $n = 1, \ldots \infty$. Writing a bit more explicitly we have for instance $\langle\delta c^4\rangle = \langle\delta c(1)\delta c(2)\delta c(3)\delta c(4)\rangle$ and a corresponding expression for $\langle\delta L^4\rangle$.

The statistical information is only seldom given *a priori* in the form of correlation functions. Usually one has to calculate the correlation functions from some other information such as "spherical or ellipsoidal elastic inclusions are embedded in a matrix material of other elastic moduli" etc. This calculation is often a most difficult task, for which the science of mathematical morphology has been developed [4, 5]. The basic equation (3.6) is sometimes named after Lippmann and Schwinger, or Dyson (i.e. LSD), who introduced such an equation into quantum mechanical scattering theory. (3.6) can be transformed into a relationship between *modified* Green's functions:

$$(3.8) \qquad \underline{\underline{\Gamma}} = (I - \underline{\underline{\Gamma}}^\circ \delta c + \underline{\underline{\Gamma}}^\circ \underline{\underline{\Gamma}}^\circ \delta c \delta c - \underline{\underline{\Gamma}}^\circ \underline{\underline{\Gamma}}^\circ \underline{\underline{\Gamma}}^\circ \delta c \delta c \delta c + \cdots - \cdots) \underline{\underline{\Gamma}}^0.$$

To calculate $\langle \underline{\underline{\Gamma}} \rangle$ from this equation we again need the statistical information on the correlation functions of the elastic moduli, $\langle \delta c^n \rangle$, $n = 1, \cdots, \infty$.

We have seen in Sec. 2, that $\langle \underline{\underline{\Gamma}} \rangle$ is the Green's function of a medium whose elasticity is described by a tensor C^{eff} of elastic moduli.

In such a medium the local elastic moduli c are uncorrelated with the distribution of the external forces F. If this is not true in some random elastic medium, then we have a more general statistical situation. In fact, to describe such a situation, we need not only to know correlations between elastic moduli at different positions, but also correlations between moduli and forces. We see this as follows: Combining (3.1) and (3.6) gives us

$$(3.9) \qquad u = \underline{\underline{\gamma}}^\circ F - \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \delta \underline{\underline{L}} F + \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \delta \underline{\underline{L}} \delta \underline{\underline{L}} F - \cdots + \cdots$$

and after averaging

$$(3.10) \qquad \langle u \rangle = \underline{\underline{\gamma}}^\circ \langle F \rangle - \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \langle \delta \underline{\underline{L}} F \rangle + \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \underline{\underline{\gamma}}^\circ \langle \delta \underline{\underline{L}} \delta \underline{\underline{L}} F \rangle - \cdots + \cdots .$$

Since we now treat a situation with $(\underline{\underline{L}} - F)$-correlations (or $(u - F)$-correlations), $\langle u \rangle$ no longer equals $\underline{\underline{\gamma}}^\circ \langle F \rangle$. To calculate $\langle u \rangle$, we rather need the mixed correlations $\langle \delta \underline{\underline{L}}^n F \rangle$, $n = 1, \cdots, \infty$, where for instance $\langle \delta \underline{\underline{L}}^3 F \rangle = \langle \delta L(1) \, \delta L(2) \, \delta L(3) \, F(4) \rangle$. If we are interested in correlation functions of u such as $\langle u(1) \, u(2) \rangle$, or $\langle u(1) \, u(2) \, u(3) \rangle$, then we need to know the set of correlation functions of the form $\langle \delta \underline{\underline{L}}^m F^n \rangle$, $m, n = 0, \ldots, \infty$, where e.g. $\langle \delta \underline{\underline{L}}^2 F^3 \rangle = \langle \delta \underline{\underline{L}}(1) \, \delta \underline{\underline{L}}(2) \, F(3) \, F(4) \, F(5) \rangle$. Here we have included the correlation functions with $m = 0$ (functions among F alone) and those with $n = 0$ (functions among $\delta \underline{\underline{L}}$ alone).

A further case: Thanks to the ergodic hypothesis, differentiation and integration commute with ensemble averaging. This means for instance, that the average of the strain tensor ε is

$$(3.11) \qquad \langle \varepsilon \rangle = \frac{1}{2}(\langle u \rangle \nabla + \nabla \langle u \rangle),$$

so that for the calculation of $\langle \varepsilon \rangle$ we need the same correlation functions like for $\langle u \rangle$, namely $\langle \delta c^n \rangle = \langle \delta c(1) \, \delta c(2) \ldots \delta c(n) \rangle$. Of course, similar statements can be made for higher derivatives of u and ε.

Summing up, we have found that for the general situation (but always lin-earized elasticity theory!) we need to know the full set of correlation functions among δL and F, or δc and F (including the correlation functions among δL (or δc) and F alone). Having this information, we can calculate, in principle, the whole statistical information about the system, again in the form of corre-lation functions, namely now the functions $\langle u^l \delta L^m F^n \rangle$ (or $\langle \varepsilon^l \delta L^m F^n \rangle$) where for instance $\langle u^2 \delta L^3 F^2 \rangle = \langle u(1)\,u(2)\delta L(3)\,\delta L(4)\,\delta L(5)\,F(6)\,F(7) \rangle$.

Its is known in the probability theory, the information about the full set of correlation functions is equivalent to that of the probability density functional. That means, given this functional, all correlation functions can be derived by well-known methods. Therefore we can formulate our general problem as follows:

Given the probability density functional $P[\delta c, F]$ in terms of δc and F, the probability density functional $P[u, \delta c, F]$ or $P[\varepsilon, \delta c, F]$ is sought for in terms of u (or ε), δc and F [1]. We think, that this formulation holds even in nonlinear elasticity.

4. Examples of applications

a. In microheterogeneous matter, the constituents which possess different elas-tic moduli usually are of different mass density (this is not true for 1-phase poly-crystals). Different mass densities imply different gravitational forces between the various constituents. The gravitational forces are clearly correlated to the dis-tribution of the masses, i.e. also of the elastic moduli. To calculate for instance the macroscopic stress state of such a medium, we need information about the (c-f)-correlations.

b. Inertia forces, like gravitational forces, are volume forces. In the case of wave propagation they are correlated with the mass distribution, hence also with the elastic moduli. To calculate problems of wave propagation, one needs again information about the (c-f)-correlations, where f are now the inertia forces. Strongly fluctuating elastic moduli cause a considerable energy flux into the *correlation waves*. E.g. correlations like $\langle uc^n \rangle$ participate in the wave motion. Here $\langle uc^3 \rangle$, for instance, stands for $\langle u(1)\,c(2)\,c(3)\,c(4) \rangle$.

c. To judge the importance of stress and strain fluctuations, their size must be known. Particularly important for this are the 2-point correlations $\langle \varepsilon(1)\,\varepsilon(2) \rangle$ and $\langle \sigma(1)\,\sigma(2) \rangle$.

d. Internal stresses (eigenstresses) appear when the incompatibility tensor does not vanish. After plastic deformation this tensor, say η, occurs as a new random function. To calculate the macroscopic eigenstress after unloading, one needs to have information about the $(\sigma - \eta)$-correlations. Since, according to a theorem of ALBENGA [6], the volume average of the eigenstresses always vanishes, one is inclined to assume that also the average strain, $\langle \varepsilon \rangle = \langle s\sigma \rangle$ vanishes. This however is not true thanks the occurrence of the $(\sigma - \eta)$-correlations.

e. A great number of further applications have to do with conditional averages. For instance, it might be of interest to know the average stress in all grains of a polycrystal, that have the same lattice orientation or grain shape. Also this problem can be solved, at least in principle, with the tools discussed in this note. We renounce further examples since those given already show the power of the method of correlation functions for the solution of statistical problems in the (linearized) elasticity theory.

5. Conclusion

For materials which are materially uniform even on the microscale, elasticity theory has proven to be a tool of extraordinary success. A large part of materials in practical use, however, are not at all materially uniform but rather heterogeneous, often on a microscale. Speaking of microscale we have in mind something that is very small compared with the scale of our perceptions – the macroscale – but distinctly larger than the atomic scale whose unit is the atomic distance. Including the atomic scale into our considerations would mean to discuss the question, to what degree can the atomic discreteness be neglected in the theory of the mechanical behaviour of micro- and macro-continuous matter. However, this topic, which is not at all scientifically exhausted, is beyond the ambition of the present note. So is also the question of nonlocality in the elastic interaction. The corresponding theory is in a rather good state, but its inclusion would enlarge this note too much.

The equations of the theory of local and linear elastic media are, of course, also valid, when the elastic parameters vary on the macro- and microscale. But the solution of these equations becomes extremely involved when the distribution of elastic parameter is partially or completely random, the case to which we now restrict ourselves.

In fact, except in very special situations, it is even impossible to write down explicitly the elastic field equations, that we had abbreviated as $\underline{L}u = F$. Even less possible is then to calculate the Green's function which solves the problem in the form $u = \underline{\underline{\gamma}}F$. All the quantities appearing in these two equations are so complex that we would need an infinity of sheets of papers to write them down. For this reason there is not even a desire to know these quantities in all detail.

It is in the spirit of probability theory and statistical mechanics, to replace the functions of the microscale by probability density functionals. These contain much less information than the original fluctuating functions. In doing this, we move from the field of continuum mechanics into the field of statistical continuum mechanics (see e.g. [7]) whose most important tool are the correlation functions. These relate to one or more random functions and can be extracted from the pertaining probability functionals. We distinguish 1-point, 2-point, 3-point, $\cdots$, n-point correlation functions, where n extends up to infinity. The great prac-

tical significance of them stems from the fact that the correlation functions of low order, e.g. $n \leq 4$, carry the most relevant information about the statistical quantities of the system. Therefore it is often good enough, to consider only a few low order correlation functions instead of their infinite set or the equivalent probability density functional.

Since the primary interest in such statistical problems usually relates to averages of relevant quantities such as stress, strain, functions of these, we need the correlation functions in order to calculate those averages. This can easily be seen using the method of Green's functions. To calculate e.g. the average strain $\langle \varepsilon \rangle$, we need the modified Green's function $\underline{\Gamma}$, or sometimes its mean $\langle \underline{\Gamma} \rangle$. They can be gained from the correlation functions of the elastic moduli, as we have shown.

Strictly speaking we have discussed the traction boundary value problem only. It is clear, however, that the displacement and mixed boundary value problem can be treated in an analogous way.

Since the appearing series developments concern multiple integrals, the solution of our problems requires a lot of numerical effort. As far as we know, correlations of 3-rd order are sometimes tractable. There is a hope that modern computers will allow us to go beyond the 3-rd order.

This note was a very brief review of an important field within the mechanical sciences. It is clear that methods analogous to those shown here can be applied also to fields other than mechanical, e.g. to electric and heat conduction, to dielectricity, magnetic phenomena and so on. Many questions are still open, in particular where mixed correlations occur. Almost nothing has been done with the *functional* theory of random elastic media.

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