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Basic inequalities for multipoint Padé approximants to Stieltjes functions

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Basic inequalities for diagonal and subdiagonal multipoint Padé approximants to N power series expansions of Stieltjes function f_0 at points x_1, x_2, ..., x_N are derived. For particular cases the inequalities obtained reduce to those obtained earlier for one-, two- and three-point Padé approximants in [1], [5] and [23], respectively. Numerical examples illustrating the relations achieved are also provided. Our results can be applied to the determination of bounds on the effective moduli of bone subjected to torsion and composites in the case of transport equations.

1. Introduction

The properties of one- two- and three-point Padé approximants to Stieltjes function, say f_0, were extensively investigated in recent years. The obtained results valid in a real domain read: (i) for $x > 0$ the sequence of the diagonal and subdiagonal one-point Padé approximants to the expansion of f_0 at $x = 0$ form upper and lower bounds uniformly converging to $f_0(x)$, cf. [1], [8] and [27]; (ii) for $x > 0$ the sequence of the diagonal two-point Padé approximants to the expansions of f_0 at $x = 0$ and $x = \infty$ also form upper and lower bounds uniformly converging to $f_0(x)$, cf. [5], [22], [23] and [9]; (iii) for $x > 0$ the sequence of the diagonal three-point Padé approximants to the expansions of f_0 at $x = 0$, $x = 1$ and $x = \infty$ form upper and lower bounds uniformly converging to $f_0(x)$, cf. [9]. The Padé approximant bounds reported in (i) - (iii) are the best ones with respect to the given number of Stieltjes series coefficients.

The main aim of this paper is to extend the validity of the inequalities for one- [1], [8], [27], two-[5], [9], [22], and three-point Padé approximants [23] to the multipoint Padé ones, constructed for power expansions of f_0 at x_1, x_2, ...,

$x_N \leq \infty$. Nontrivial practical applications of the inequalities obtained previously to mechanical problems are presented in [21], [24], [26] and [27].

This paper is organized as follows: In Sec. 2 we introduce the basic definitions, notations and assumptions dealing with Stieltjes functions, multipoint Padé approximants and multipoint continued fractions. The basic inequalities for diagonal and subdiagonal multipoint Padé approximants to a Stieltjes function are derived in Sec. 3. In Sec. 4 illustrative examples are presented. Particular cases of the inequalities obtained are discussed in Sec. 5. In Sec. 6 the practical applications of the multipoint Padé approximants are presented. The results achieved are summarized in Sec. 7

2. Preliminaries

Let us consider a real function f analytic at N different points x_1, x_2, ..., x_N, where without loss of generality we assume

$$x_1 < x_2 < \ldots < x_N. \tag{2.1}$$

The power expansions of f at the above points are

$$f(x) = \sum_{k=0}^{\infty} c_k(x_j)(x - x_j)^k, \quad j = 1, 2 \ldots, N. \tag{2.2}$$

In practical situations we know only a few first coefficients of each expansion (2.1) and then we have to deal with the limited information characterized by the truncated power series

$$f(x) = \sum_{k=0}^{p_j - 1} c_k(x_j)(x - x_j)^k + O((x - x_j)^{p_j}), \quad j = 1, 2, \ldots, N, \tag{2.3}$$

where p_j denote the given number of coefficients of power series at each point.

Now we are in a position to recall from [2, Chap.8] the definition of multipoint Padé approximants to f.

DEFINITION 1. *The N-point Padé approximant to a function f, corresponding to the expansions (2.3), if it exists, is a rational function P_m/Q_n denoted by $[m/n]_f^{N,p}$,*

$$\begin{gathered}[m/n]_f^{N,p} = \frac{P_m(x)}{Q_n(x)} = \frac{a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_m x^m}{1 + b x^1 + b_2 x^2 + \cdots + b_n x^n},\\ p = \sum_{j=1}^{N} p_j, \quad m + n = p - 1,\end{gathered} \tag{2.4}$$

satisfying the following relations:

$$f(x) - [m/n]_f^{N,p}(x) = O((x - x_j)^{p_j}), \qquad j = 1, 2, ..., N. \tag{2.5}$$

In the following we deal only with diagonal and subdiagonal approximants, then

$$m = p - 1 - E\left(\frac{p-1}{2}\right), \quad n = E\left(\frac{p-1}{2}\right), \tag{2.6}$$

where $E(x)$ denotes the integer part of x.

As usual, for the ordinary one-point Padé approximants the relations (2.5), where f and $[m/n]$ must be understood as the power series expansions at the corresponding points x_j, represent a linear system defining the coefficients $(a_0, a_1, ..., a_m, b_1, b_2, ..., b_n)$.

REMARK 1. All previous definitions are also valid for analytic functions of complex variable and for points x_j in the complex domain.

In our investigations we are concerned with Stieltjes functions and the N-point Padé approximants to the corresponding Stieltjes series. Let f_0 be a Stieltjes function having the following representation, see [1], [9]

$$\begin{gathered} f_0(x) = f_0(0) + x \int_0^{1/\rho} \frac{d\gamma_0(u)}{1 + xu}, \\ \lim_{x \to 0} f_0(x) = f_0(0) \equiv g_0 > 0, \quad \lim_{x \to \infty} f_0(x) = d_0 < \infty, \end{gathered} \tag{2.7}$$

where the spectrum γ_0 is a real, bounded non-decreasing function.

Let us introduce the following notation:

$$\mathop{\mathbf{K}}_{k=1}^{r} \frac{a_k}{1} := \cfrac{a_1}{1 + \cfrac{a_2}{1 + \ddots \cfrac{a_r}{1.}}} \quad \text{and} \quad g_k := f_k(x_j). \tag{2.8}$$

Functions f_k are defined in the following procedure of expansion of f_0 in one-point (afterwards: in N-point) continued fraction

$$\begin{aligned} f_0(x) = f_0(x_1) + (x - x_1) f_1(x) &= f_0(x_1) + \frac{(x - x_1) f_1(x_1)}{1 + (x - x_1) f_2(x)} = ... \\ &= f_0(x_1) + \mathop{\mathbf{K}}_{k=1}^{r-1} \frac{(x - x_1) f_k(x_1)}{1} + \frac{(x - x_1) f_r(x)}{1} \\ &\equiv g_0 + \mathop{\mathbf{K}}_{k=1}^{r-1} \frac{(x - x_1) g_k}{1} + \frac{(x - x_1) f_r(x)}{1}. \end{aligned} \tag{2.9}$$

All functions f_k are also Stieltjes functions.

Now we focus our attention on N-point continued fraction expansion of f_0 at the points x_1, x_2, ..., x_N. Let us introduce the non-decreasing step-wise function L

(2.10)
$$L(x) = \sum_{j=1}^{N} p_j H(x - x_j), \quad p = \sum_{j=1}^{N} p_j = L(x_N), \quad H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 71 & \text{if } t \geq x. \end{cases}$$

The value $L(x)$ denotes the total number of given coefficients of power expansions of f_0 at all points $x_j \leq x : L(x_k) = p_1 + p_2 + \ldots + p_k$.

To construct the multipoint continued fraction, we begin by expanding f_0 at x_1 as shown in (2. 9), we follow by expanding $f_{L(x_1)}$ at x_2, and so on:

$$f_0(x) = g_0 + \mathop{\mathbf{K}}_{k=1}^{L(x_1)-1} \frac{(x-x_1)g_k}{1} \; \frac{x-x_1}{+\, x-x_2} \mathop{\mathbf{K}}_{k=L(x_1)}^{L(x_2)-1} \frac{(x-x_2)g_k}{1} \cdots \frac{(x-x_{N-1})}{+\;(x-x_N)} \mathop{\mathbf{K}}_{k=L(x_{N-1})}^{L(x_N)-1} \frac{(x-x_N)g_k}{1} \; \frac{(x-x_N)f_p(x)}{+\quad 1}. \tag{2.11}$$

One can easily verify that the N-point Padé approximant $[m/n]_{f_0}^{N,p}$ to f_0 is equal to the following truncated continued fraction:

$$m/n]_{f_0}^{N,p}(x) = g_0 + \mathop{\mathbf{K}}_{k=1}^{L(x_1)-1} \frac{(x-x_1)g_k}{1} \; \frac{x-x_1}{+\, x-x_2} \mathop{\mathbf{K}}_{k=L(x_1)}^{L(x_2)-1} \frac{(x-x_2)g_k}{1} \cdots \frac{(x-x_{N-1})}{+\;(x-x_N)} \mathop{\mathbf{K}}_{k=L(x_{N-1})}^{L(x_N)-1} \frac{(x-x_N)g_k}{1}, \tag{2.12}$$

where, because f_k are positive functions,

$$\forall k : g_k > 0. \tag{2.13}$$

3. Basic inequality

THEOREM 1. *Let f_0 be a Stieltjes function (2. 7) and let N power series (2. 2) have nonzero radii of convergence; then the diagonal and subdiagonal N-point Padé approximants $[m/n]_{f_0}^{N,p}$ obey the following inequality:*

$$x \in]-\rho, \infty[: \quad (-1)^{L(x)} \, [m/n]_{f_0}^{N,p}(x) \geq (-1)^{L(x)} \, f_0(x), \tag{3.1}$$

where m and n are defined by (2. 6) and the function L by (2. 10).

S k e t c h o f t h e p r o o f. Let us start from the Stieltjes function f_0 represented by the following multipoint continued fraction (one step less than in (2.11))

$$(3.2)\quad f_0(x) = g_0 + \mathop{\mathrm{K}}_{k=1}^{L(x_1)-1} \frac{(x-x_1)g_k}{1} + \frac{(x-x_1)}{x-x_2} \mathop{\mathrm{K}}_{k=L(x_1)}^{L(x_2)-1} \frac{(x-x_2)g_k}{1} \dots + \frac{x-x_{N-1}}{x-x_N} \mathop{\mathrm{K}}_{k=L(x_{N-1})}^{L(x_N)-2} \frac{(x-x_N)g_k}{1} + \frac{(x-x_N)f_{L(x_N)-1}(x)}{1}.$$

For simplicity, choosing $x_1 = 0$ and replacing $f_{L(x_N)-1}(x)$ by $g_{L(x_N)-1} := f_{L(x_N)-1}(x_N)$, one obtains the following Padé approximant $[m/n]_{f_0}^{N,p}$ to f_0 :

$$(3.3)\quad [m/n]_{f_0}^{N,p}(x) = g_0 + \mathop{\mathrm{K}}_{k=1}^{L(0)-1} \frac{xg_k}{1} + \frac{x}{x-x_2} \mathop{\mathrm{K}}_{k=L(0)}^{L(x_2)-1} \frac{(x-x_2)g_k}{1} \dots + \frac{x-x_{N-1}}{x-x_N} \mathop{\mathrm{K}}_{k=L(x_{N-1})}^{L(x_N)-2} \frac{(x-x_N)g_k}{1} + \frac{(x-x_N)g_{L(x_N)-1}}{1}.$$

Since all f_k are Stieltjes functions, they are decreasing positive functions on $]-\rho,\infty[$, and then the following inequalities hold:

$$(3.4)\qquad 1+(x-x_j)g_k \geq 0, \qquad j=1,2,...,N$$

and

$$(3.5)\qquad x\in[\,-\rho,\infty[: \qquad (x-x_N)f_{L(x_N)-1}(x) \leq (x-x_N)g_{L(x_N)-1}.$$

The recurrence formulae for (3.2) and (3.3) jointly with the relations (2.12) and (3.5), yield the basic inequality (3.1). □

In the next section the numerical example illustrating the universality of the inequality (3.1) will be presented.

4. Illustrative examples

Let us start our considerations from the following Stieltjes function:

$$(4.1)\qquad f_0(x) = 1+\ln(0.5(x+1))$$

having at $x = 0$, 1, 10^3 and 10^6 the truncated power expansions

$$(4.2)\qquad \begin{aligned} f_0(x) &= 0.307 + O(x),\\ f_0(x) &= 1+0.500(x-1)-0.125(x-1)^2+O((x-1)^3),\\ f_0(x) &= 7.21+10^{-3}(x-10^3)+O((x-10^3)^2),\\ f_0(x) &= 14.1+10^{-6}(x-10^6)+O((x-10^6)^2). \end{aligned}$$

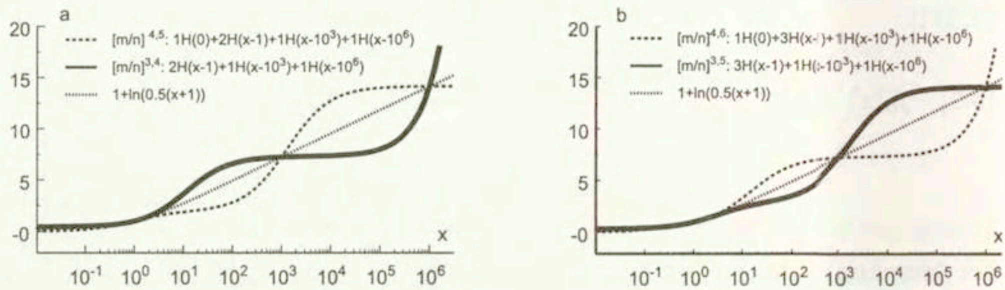

FIG. 1. Three- and four-points Padé approximants to the Stieltjes function $f_0(x) = 1 + \ln(0.5(x + 1))$, see Theorem 1.

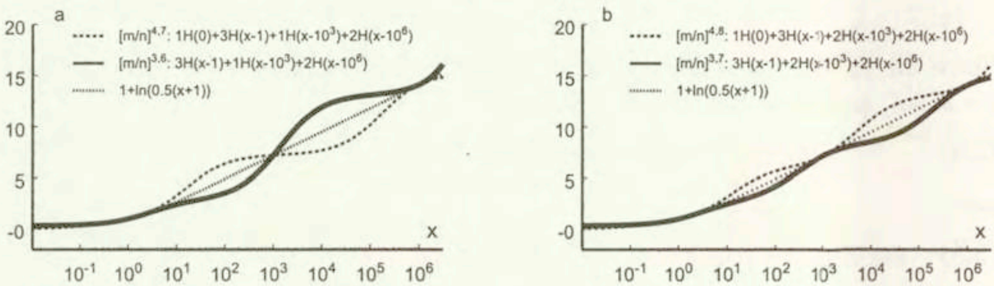

FIG. 2. Three- and four-points Padé approximants to the Stieltjes function $f_0(x) = 1 + \ln(0.5(x + 1))$, see Theorem 1.

Evaluated from the input data (4. 2), the four-point Padé approximant $[m/n]^{4,4}_{f_0}$, $L(x) = 1H(x) + 1H(x-1) + 1H(x-10^3) + 1H(x-10^6)$, to $1 + \ln, (0.5(x + 1))$ takes the form

$$(4.3) \qquad [m/n]^{4,4}_{f_0} = 0.307 + \frac{0.693x}{1} + \frac{0\ .099(x-1)}{1} + \frac{0\ .983 \times 10^{-6}(x-10^3)}{1}.$$

The discrete values of $L(x)$, $(-1)^{L(x)} f_0(x)$ and $(-1)^{L(x)}[m/n]^{4,4}_{f_0}(x)$ taken at $x = -0.5,\ 0.5,\ 500,\ 5 \times 10^5,\ 10^8$ are gathered in Table 1. The functions $(-1)^{L(x)} f_0(x)$ and $(-1)^{L(x)}[m/n]^{4,4}_{f_0}$ (see Table 1) satisfy the inequality (3. 1). Also the values of $[m/n]^{N,p}_{f_0}$ to $1 + \ln(0.5(x + 1))$ presented in Figs. 1 and 2 obey the basic inequality (3. 1).

Table 1. The input data $(-1)^{L(x)} f_0(x)$ and $(-1)[m/n]_{f_0}^{4,4}$ for numerical verification of the basic inequality (3. 1), where $f_0(x) = 1 + ln(0.5(x+1))$ and $L(x) = 1H(x) + 1H(x-1) + 1H(x-10^3) + 1H(x-10^6)$.

x	-0.5	0.5	500	5×10^4	10^8
$L(x)$	0	1	2	3	4
$(-1)^{L(x)} f_0(x)$	-0.386	-0.712	6.52	-11.1	18.7
$(-1)^{L(x)} [m/n]_{f_0}^{4,4}(x)$	-0.100	-0.672	7.15	-7.61	692

5. Particular cases of the basic inequality

In this section we will discuss the particular cases of the basic inequality (3. 1). We start from one-point Padé approximants $[m/n]_{f_0}^{1,p_0}$ to Stieltjes function f_0. For such a case, Theorem 1 reduces to (we recall our choice $x_1 = 0$)

COROLLARY 1. Diagonal and subdiagonal one-point Padé approximants $[m/n]_{f_0}^{1,p_1}$ to power series of Stieltjes

$$f_0(x) = \sum_{k=0}^{\infty} c_k(0) x^k \tag{5.1}$$

with $L(x) = p_1 H(x)$ obey the following inequality:

$$(-1)^{p_1 H(x)} \, [m/n]_{f_0}^{1,p_1}(x) \ \geq (-1)^{p_1 H(x)} \, f_0(x) \quad \text{in} \quad -\rho < x < \infty. \tag{5.2}$$

For diagonal and subdiagonal one-point Padé approximants the Corollary 1 coincides with Theorem 15.2 proved by BAKER in his book [1].

For two-point Padé approximants $[m/n]_{f_0}^{2,p}$ to the Stieltjes function f_0, the basic Theorem 1 is formulated as follows.

COROLLARY 2. Diagonal and subdiagonal two-point Padé approximants $[m/n]_{f_0}^{2,p}$ to the series of Stieltjes

$$f_0(x) = \sum_{k=0}^{\infty} c_k(0) x^k, \quad f_0(x) = \sum_{k=0}^{\infty} c_k(x_2)(x - x_2)^k \tag{5.3}$$

with $L(x) = p_1 H(x) + p_2 H(x - x_2)$ obey the following inequality:

$$(-1)^{p_1 H(x) + p_2 H(x-x_2)} \, [m/n]_{f_0}^{2,p}(x) \geq (-1)^{p_1 H(x) + p_2 H(x-x_2)} \, f_0(x) \tag{5.4}$$

$$\text{in} \quad -\rho \leq x \leq \infty.$$

Further, if $x_2 \to \infty$ then from Corollary 2 we infer

COROLLARY 3. Let us denote by $[m/n]^{2,p}_{f_0,x_2}$ the two-point Padé approximants at the points $x_1 = 0$ and x_2 to the series of Stieltjes

$$f_0(x) = \sum_{k=0}^{\infty} c_k(0)x^k, \quad f_0(x) = \sum_{k=0}^{\infty} c_k(x_2)(x - x_2)^k \tag{5.5}$$

with $L(x) = p_1 H(x) + p_2 H(x - x_2)$. For $x_2 \to \infty$ these diagonal and subdiagonal two-point approximants have finite limits denoted as follows

$$\lim_{x_2 \to \infty} [m/n]^{2,p}_{f_0,x_2}(x) = [m/n]^{2,p}_{f_0,\infty}(x). \tag{5.6}$$

Moreover, in $-\rho \leq x \leq \infty$ the following inequalities are satisfied:

$$(-1)^{p_1 H(x)} \, [m/n]^{2,p}_{f_0,\infty}(x) \; \geq (-1)^{p_1 H(x)} \, f_0(x). \tag{5.7}$$

Corollary 3 coincides with Corollaries 4.6 and 4.7 proved by A. BULTHEEL *et al.* in [5] and also with Theorems 5.1 and 5.2 reported by TOKARZEWSKI and TELEGA in [26].

Let us discuss the three-point Padé approximants $[m/n]^{3,p}_{f_0,x_3}$ to f_0 for the case $x_3 \to \infty$. Then Theorem 1 yields:

COROLLARY 4. For $x_3 \to \infty$, the diagonal and subdiagonal three-point Padé approximants $[m/n]^{3,p}_{f_0,x_3}$ to the series of Stieltjes

$$\begin{aligned} f_0(x) &= \sum_{k=0}^{\infty} c_k(0)x^k, \\ f_0(x) &= \sum_{k=0}^{\infty} c_k(x_2)(x - x_2)^k, \\ f_0(x) &= \sum_{k=0}^{\infty} c_k(x_3)(x - x_3)^k, \end{aligned} \tag{5.8}$$

with $L(x) = p_1 H(x) + p_2 H(x - x_2) + p_3 H(x - x_3)$, obey in $-\rho \leq x \leq \infty$ the following inequality:

$$(-1)^{p_1 H(x) + p_2 H(x - x_2)} [m/n]^{3,p}_{f_0,\infty}(x) \geq (-1)^{p_1 H(x) + p_2 H(x - x_2)} f_0(x), \tag{5.9}$$

where

$$\lim_{x_3 \to \infty} [m/n]^{3,p}_{f_0,x_3}(x) = [m/n]^{3,p}_{f_0,\infty}(x). \tag{5.10}$$

Corollaries 4 and 3 coincide with Theorem 10.1 derived by TOKARZEWSKI in [23].

6. Padé approximants application

PERRINS *et al.* [19] investigated numerically and experimentally the effective conductivity ε_{ef} of hexagonal array of cylinders embedded in an infinite matrix (see Fig. 3) as a function of φ and $h = \varepsilon_2/\varepsilon_1$, where φ, ε_1 and ε_2 denote the volume fraction of cylinders and the conductivities of the matrix and inclusions, respectively. It is well known that ε_{ef} has a Stieltjes integral representation, cf. [4] and [10]

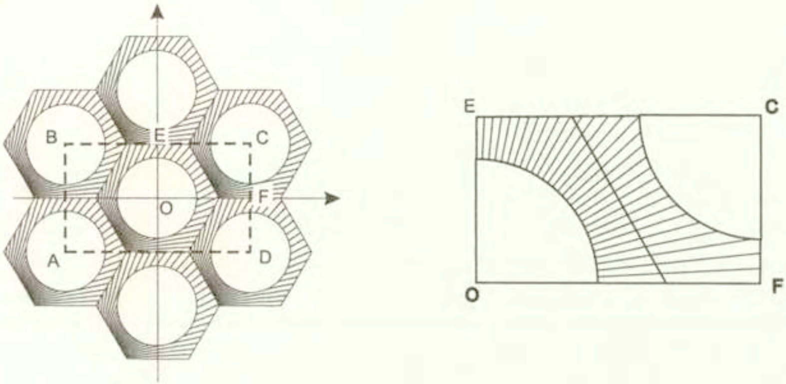

FIG. 3. Hexagonal array of cylinders embedded in an infinite matrix; ABCD-the unit cell, OECF-the sub-unit cell.

(6.1)
$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = \frac{\varepsilon_{ef}(0)}{\varepsilon_1} + h\int_0^\infty \frac{d\gamma(u)}{1+hu}.$$

From (6.1) it follows that Padé approximants to power expansions of $\varepsilon_{ef}/\varepsilon_1$ satisfy the basic inequality (3.1).

Now we are in a position to recall the experiment performed by PERRINS *et al.* [19], see Fig. 4. The measurements of $\varepsilon_{ef}(h)/\varepsilon_1$ elaborated by these authors allowed to construct the following power expansions of $\varepsilon_{ef}(h)/\varepsilon_1$:

(i) For $\varphi = 0.65$:

(6.2)
$$\begin{aligned}
\frac{\varepsilon_{ef}(h)}{\varepsilon_1} &= 0.203 + O(h),\\
\frac{\varepsilon_{ef}(h)}{\varepsilon_1} &= 1 + 0.65(h-1) - 0.114(h-1)^2 + O(h-1)^3,\\
\frac{\varepsilon_{ef}(h)}{\varepsilon_1} &= 4.93 + O(1/h).
\end{aligned}$$

(ii) For $\varphi = 0.76$:

$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 0.132 + O(h),$$

(6.3)
$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 1 + 0.76(h-1) - 0.091(h-1)^2 + O(h-1)^3,$$

$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 7.58 + O(1/h).$$

(iii) For $\varphi = 0.80$:

$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 0.203 + O(h),$$

(6.4)
$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 1 + 0.80(h-1) - 0.080(h-1)^2 + O(h-1)^3,$$

$$\frac{\varepsilon_{ef}(h)}{\varepsilon_1} = 10.34 + O(1/h).$$

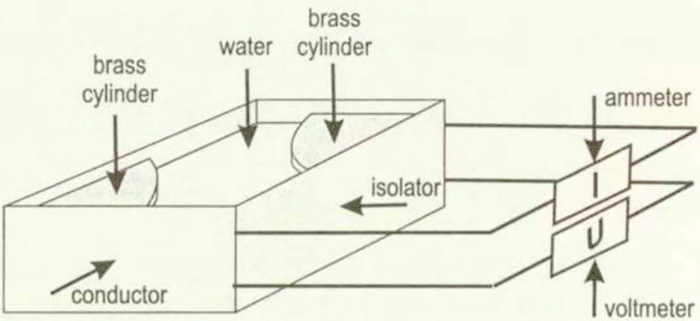

FIG. 4. Scheme of experiment performed by Perrins *et al.* [19] for the sub-unit cell of hexagonal array of cylinders, cf. Fig. 3.

The coefficients of (6.2), (6.3) and (6.4) are obtained by experimental measurements of the electrical potentials and the electrical currents appearing in the two-phase system shown in Fig. 4.

The multipoint Padé approximants $[2/2]^{3,4}$ (solid line) and $[1/1]^{1,2}$ (dashed lines) have been evaluated to "experimental" power expansions of $\varepsilon_{ef}/\varepsilon_1$ represented by series (6.2)-(6.4), see Fig. 5. According to Theorem 1 those series form the upper and lower bounds on the effective conductivity $\varepsilon_{ef}/\varepsilon_1$ of hexagonal array of cylinders, respectively. The theoretical conductivity $\varepsilon_{ef}/\varepsilon_1$ evaluated in [19] is also shown in Fig. 5 for comparison.

7. Final remark

A new basic inequality (3.1) for diagonal and subdiagonal multipoint Padé approximants $[m/n]_{f_0}^{N,p}$ to a Stieltjes function $f_0(x)$, $-\rho \leq x \leq \infty$, has been derived. This inequality generalizes the previous ones formulated for one-, two-, and three- point Padé approximants $[m/n]_{f_0}^{N,p}$ to f_0.

The transport coefficients such as thermal conductivity, magnetic permeability, dielectric constant, diffusion coefficient, have Stieltjes integral representation. On account of that, inequality (3.1) yields upper and lower bounds on the transport moduli of macroscopically isotropic two-phase media, cf. [24] and [27]. They are the most general bounds reported until now in both the mathematical and mechanical literature and can also be used to the study of bone torsion, cf. [25].

Multipoint Padé approximants approach presented in this paper is particular useful for the prediction of the effective moduli of inhomogeneous two-phase media from both the theoretical and experimental data, cf. Fig. 5.

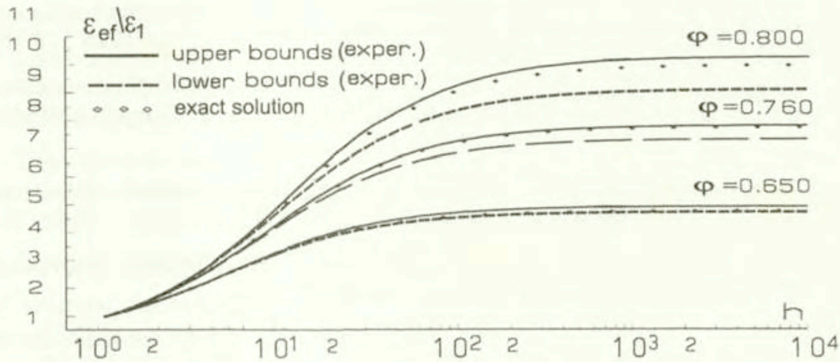

FIG. 5. The multipoint Padé approximants $[2/2]^{3,4}$ (solid line), $[1/1]^{1,2}$ (dashed lines) to the "experimental" power expansions of $\varepsilon_{ef}/\varepsilon_1$ given by (6.2)-(6.4) and the theoretical power expansion of $\varepsilon_{ef}/\varepsilon_1$ (scattered line) reported in [19].

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References

1. G. A. BAKER JR., *Essentials of Padé approximants*, Acad. Press, London 1975.

2. G. A. BAKER JR. and P. GRAVES-MORRIS, *Padé approximants*, Cambridge University Press, second edition, 1996.

3. D. J. BERGMAN, *The dielectric constant of a composite material- A problem in classical physics*, Phys. Report., **C43**, 377-407, 1978.

4. D. J. BERGMAN, *Bounds for the complex dielectric constant of a two-component composite material*, Phys. Rev. B, **23**, 3058-3065, 1981.

5. A. BULTHEEL, P. GONZÁLES-VERA and R. ORIVE, *Quadrature on the half-line and two-point Padé approximants to Stieltjes functions. Part I. Algebraic aspects*, J. of Comp. Appl. Math, **65**, 57-72, 1995.

6. J. GILEWICZ, *Approximants de Padé*, Lecture Notes in Mathematics vol. **667**, Springer Verlag, 1978.

7. J. GILEWICZ and A. P. MAGNUS, *Inverse Stieltjes iterates and errors of Padé approximants in the whole complex plane*, J. Comput. Appl. Math. **49**, 79-84, 1993.

8. J. GILEWICZ and A. P. MAGNUS, *Optimal inequalities of Padé approximants errors in the Stieltjes case: closing result*, Integral Transforms and Special Functions **1**, 9-18, 1993.

9. J. GILEWICZ, M. PINDOR, J. J. TELEGA and S. TOKARZEWSKI, *Continued fractions, two-point Padé approximants and errors in the Stieltjes case*, J. Comp. Appl. Math. (to appear).

10. K. GOLDEN and G. PAPANICOLAOU, *Bounds for effective parameters of heterogeneous media by analytic continuation*, Com. Math. Phys., **90**, 473-491, 1983.

11. P. GONZALEZ-VERA and O. NJÅSTAD, *Convergence of two-point Padé approximants to series of Stieltjes*, J. of Comp. Appl. Math., **32**, 97-105, 1990.

12. W. B. GRAGG, *Truncation error bounds for T-fractions*, [in:] Approximation Theory III, E. W. CHENEY [Ed.] 455-460, Academic Press, New York 1980.

13. Z. HASHIN and S. SHTRIKMAN, *A variational approach to the theory of the effective magnetic permeability of multiphase materials* J. Appl. Phys. Sol., **33**, 3125-3131, 1962.

14. W. B. JONES, O. NJÅSTAD and W. J. THRON, *Two-point Padé expansions for a family of analytic functions*, J. Comp. Appl. Math., **9**, 105-123, 1983.

15. W. B. JONES and W. J. THRON, *Continued fraction analytic theory and its applications*, Encyclopedia of Mathematics and its Applications, Vol. **11**, Addison-Wesley Publ. Co., London 1980.

16. L. LORENTZEN and H. WAADELAND, *Continued Fraction with Applications*, [in:] Studies Computational Mathematics 3, C. BREZINSKI and L. WUYTACK [Eds.] Elsevier Science Publishers 1992.

17. G. W. MILTON, *Bounds on the complex permittivity of a two-component composite material*, J. Appl. Phys., **52**, 5286-5293, 1981.

18. G. W. MILTON, *Bounds on the transport and optical properties of a two- component composite material*, J. Appl. Phys., **52**, 5294-5304, 1981.

19. W. T. PERRINS, D. R. MCKENZIE and R. C. MCPHEDRAN, *Transport properties of regular array of cylinders*, Proc. R. Soc. Lond. A **369**, 207-225, 1979.

20. R. C. MCPHEDRAN and G. W. MILTON, *Bounds and exact theories for the transport properties of inhomogeneous media,* Appl. Phys. A, **26**, 207-220, 1981.

21. J. J. TELEGA, S. TOKARZEWSKI and A. GAŁKA, *Modelling torsional properties of human bones by multipoint Padé approximants,* [in:] Numerical Analysis and its Application, L. VULKOV, J. WAŚNIEWSKI and P. YALAMOV [Eds.], 741-748, Springer-Verlag, Berlin 2001.

22. S. TOKARZEWSKI, *Two-point Padé approximants for the expansion of Stieltjes functions in a real domain,* J. Comp. Appl. Math, **67**, 59-72, 1996.

23. S. TOKARZEWSKI, *N-point Padé approximants to real valued Stieltjes series with nonzero radii of convergence,* J. Comp. Appl. Math, **75**, 259-280, 1996.

24. S. TOKARZEWSKI, A. GAŁKA and I. ANDRIANOV, *Bounds on the effective transport coefficients of two-phase media from discrete theoretical and experimental data,* Comp. Assis. Mech. and Eng. Sc., **4**, 229- 241, 1997.

25. S. TOKARZEWSKI, J. J. TELEGA and A. GAŁKA, *Torsional rigidities of cancellous bone filled with marrow: The application of multipoint Padé approximants,* Engineering Transactions **49**, 135-153, 2001.

26. S. TOKARZEWSKI and J. J. TELEGA, *Inequalities for two-point Padè approximants to the expansions of Stieltjes functions in a real domain,* Acta Applicandae Mathematicae, **48**, 285-297, 1997.

27. S. TOKARZEWSKI and J. J. TELEGA, *Bounds on effective moduli by analytical continuation of the Stieltjes function expanded at zero and infinity,* Z. angew. Math. Phys. **48**, 1-20, 1997.

28. H. S. WALL, *Analytic theory of continued fractions,* Van Nostrand, New York 1948.

29. O. WIENER, *Abhandlungen der mathematisch-physischen Klasse der Könighlichen Sachsischen Gesellschaft der Wissenschaften,* Abh. Sächs. Akad. Wiss. Leipzig Math.- Naturwiss. Kl., **32**, 509, 1912.

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