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A domain of influence theorem in linear thermo-elasticity with thermal relaxation and internal variable

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A DOMAIN OF INFLUENCE theorem is proved for a linear thermoelastic solid with a Cattaneo's type heat conduction law and a scalar internal variable. The obtained result is applied to prove the hyperbolicity of a semiempirical heat conduction theory, describing the propagation of thermal waves in crystals at low temperatures.

1. Introduction

A DOMAIN OF INFLUENCE theorem is one of the basic results of classical isothermal elastodynamics [1, 2]. It asserts that for a finite time $t > 0$ a solution of a given initial and boundary value problem, corresponding to the data which are defined in a bounded support, vanishes outside a bounded domain $D(t)$. Its physical interpretation is that an initial perturbation of a bounded elastic domain gives rise to an elastic disturbance which for any $t > 0$ cannot occupy the whole space, i. e. it propagates with a finite speed. Such a theorem cannot be proved in classical linear coupled thermoelasticity since the Fourier law of heat conduction implies an infinite speed of thermal disturbances [3]. LORD and SHULMAN [4] proposed a generalized dynamical theory of thermoelasticity which is based on a generalized heat conduction law [3]. Some domain of influence theorems in the framework of the afore-mentioned theory has been proved in [5 - 8]. Different authors [9 - 14] approach the problem of finite speed of thermal waves by introducing into the constitutive equation some additional internal variables related to the thermal inertia of the body at hand. The ensuing initial and boundary value problems are transformed from the mixed hyperbolic-parabolic type to the hyperbolic type. This situation is typical of the internal variables theory since very often additional variables are introduced to eliminate the paradox of infinite speed of propagation [15, 16]. For linear and quasi-linear systems of evolution equations the hyperbolicity is assured if a given matrix, related to the coefficients of the system, admits real eigenvalues and a corresponding complete set of eigenvectors spanning the space of states [17, 18]. However, this property in general cannot be assured by *a priori* conditions but very often these conditions depend on the solution itself. In other words, these should be regarded as compatibility conditions in the sense that those solutions which do not verify them do not lead to a finite speed of propagation of thermomechanical disturbances. In many cases the notion of domain of influence yields the possibility of establishing a priori conditions ensuring the hyperbolicity. This renders the study of wave propagation more perspicuous.

In this paper we prove a domain of influence theorem for a linear thermoelastic solid obeying a generalized heat conduction equation and with an additional scalar internal variable. The physical nature of this variable will be not specified, allowing thus the model to describe a wide class of phenomena. In Sec. 2 we specify the model and its thermodynamic properties. In Sec. 3, after posing the initial and boundary value problem we are faced with, we enunciate our main hypotheses on the data and on the material functions. Then, in Sec. 4, we apply the technique developed in [8] to establish *a domain of dependence inequality* which, in Sec. 5, is used to prove our main result, i.e. a domain of influence theorem. Finally, in Sec. 6, the obtained result is applied to the so-called *semi-empirical heat conduction model*, introduced by Kosiński and co-workers [10, 12, 19], to describe non-Fourier heat conduction in solids.

2. The physical model

Let us consider a linear thermoelastic body B which is identified with an open and connected region C of the Euclidean three-dimensional point space E_3. The set C is supposed to be regular and, generally, unbounded. The fundamental system of equations governing the time evolution of B consists of:

1. The equation of motion

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\mathbf{S} + \mathbf{b}, \tag{2.1}$$

   where ρ is the mass density, $\mathbf{u}$ the field of displacement, $\mathbf{S}$ the stress tensor, $\mathbf{b}$ the body force;

2. The balance of energy

$$\rho\dot{\epsilon} = \mathbf{S}\cdot\dot{\mathbf{E}} - \operatorname{div}\mathbf{q} + \rho\mathrm{r}, \tag{2.2}$$

   where ϵ means the specific internal energy, $\mathbf{E} \equiv \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain tensor, $\mathbf{q}$ the heat flux vector and r the radiating heat supply;

3. The second law of thermodynamics, expressed by the Clausius-Duhem inequality

$$\rho\dot{\eta} + \operatorname{div}\frac{\mathbf{q}}{\theta} - \rho\frac{r}{\theta} \geq 0, \tag{2.3}$$

where η is the specific entropy and θ the absolute temperature;

4. The Maxwell-Cattaneo heat conduction equation

$$t_0\dot{\mathbf{q}} + \mathbf{q} = -\mathbf{K}\nabla\theta, \tag{2.4}$$

where constant t_0 is a suitable relaxation time and $\mathbf{K}$ the conductivity tensor;

5. The evolution equation for a temperature-dependent scalar internal variable α which in the present paper is set in the form

$$\dot{\alpha} = m\theta + n\alpha. \tag{2.5}$$

The scalar functions m and n express suitable material properties whose physical nature will remain unspecified. However, we observe that at the equilibrium ($\dot{\alpha} = 0$), α is proportional to the absolute temperature.

Taking into account (2.2), inequality (2.3) may be set in the form

$$-\rho(\dot{\Psi} + \eta\dot{\theta}) + \mathbf{S}\cdot\dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta \geq 0, \tag{2.6}$$

where

$$\Psi \equiv \epsilon - \theta\eta \tag{2.7}$$

is the Helmholtz free energy. Finally, a set of constitutive equations of the type

$$\Phi = \Phi^*(\theta, \nabla\theta, \mathbf{E}, \alpha, \nabla\alpha), \tag{2.8}$$

where Φ is an element of the set $\{\mathbf{S}, \epsilon, \Psi, \mathbf{q}\}$, characterizes the model under analysis. Compatibility of (2.8) with (2.6) implies the thermodynamical restrictions [20]

$$\eta = -\frac{\partial\Psi}{\partial\theta}, \tag{2.9}$$

$$\mathbf{S} = \rho\frac{\partial\Psi}{\partial\mathbf{E}}, \tag{2.10}$$

$$\Psi = \Psi(\theta, \mathbf{E}, \alpha, \nabla\alpha), \tag{2.11}$$

$$A\dot{\alpha} + \tilde{\mathbf{A}}\nabla\dot{\alpha} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta \geq 0, \tag{2.12}$$

where $A \equiv \rho \dfrac{\partial \Psi}{\partial \alpha}$ is the so-called *affinity* representing the generalized force conjugated to α and $\tilde{A} \equiv \rho \dfrac{\partial \Psi}{\partial \nabla \alpha}$ [21, 22].

We pursue our analysis under the additional hypotheses that the stress tensor does not depend on α or on $\nabla\alpha$ and the internal energy does not depend on $\nabla\alpha$.

In such a case, the linearization procedure [20] leads to the following constitutive equation for $\mathbf{S}$:

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] + \theta \mathbf{M}, \tag{2.13}$$

where $\mathbf{C}$ is the classical fourth order tensor of linear elasticity while $\mathbf{M}$ is a second order symmetric tensor accounting for the stress-temperature relation. From (2. 13) and (2. 2) we get

$$c_e \dot{\theta} + c_\alpha \dot{\alpha} = \theta_0 \mathbf{M} \cdot \dot{\mathbf{E}} - \mathrm{div}\mathbf{q} + \rho \mathrm{r}, \tag{2.14}$$

where $c_e \equiv \rho \dfrac{\partial \epsilon}{\partial \theta}$ is the heat capacity per unit of volume and $c_\alpha \equiv \rho \dfrac{\partial \epsilon}{\partial \alpha}$ is the latent heat per unit of volume related to the presence of the internal variable. Moreover, (2. 10) and (2. 13) yield

$$\Psi(\theta, \mathbf{E}, \alpha) = \frac{1}{2\rho} \mathbf{E} \cdot \mathbf{C}[\mathbf{E}] + \frac{1}{\rho} \theta \mathbf{M} \cdot \mathbf{E} + \Psi_0(\alpha, \theta, \nabla\alpha), \tag{2.15}$$

where Ψ_0 is the free energy corresponding to a rigid motion.

3. The initial and boundary value problem: basic assumptions and postulates

The fields ρ, c_e, c_α, m, n, $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ represent the material thermomechanical properties of B and are supposed to be prescribed. The mass density ρ and the specific and latent heats c_e and c_α, respectively, and the parameter m are assumed to be positive fields on $\bar{C}$, i.e.

$$m > 0, \quad \rho > 0, \quad c_e > 0, \quad c_\alpha > 0 \quad \text{on } \bar{C}. \tag{3.1}$$

Requirements of stability for the solution of (2. 5) force n to be negative.

The elasticity tensor is symmetric and positive semidefinite so that [1)]

$$\mathbf{A} \cdot \mathbf{C}[\mathbf{B}] = \mathbf{B} \cdot \mathbf{C}[\mathbf{A}], \forall\, \mathbf{A}, \mathbf{B} \in Lin, \tag{3.2}$$

$$\mathbf{A} \cdot \mathbf{C}[\mathbf{A}] \geq 0 \ \ \forall \mathbf{A} \in Lin. \tag{3.3}$$

[1)] We denote by V the basic Euclidean three-dimensional vector space and by Lin the nine-dimensional vector space of all linear mappings from V to V (second order tensors on V).

Analogous relations are required to be satisfied by $\mathbf{K}$ and $\mathbf{M}$:

$$\mathbf{a}\cdot\mathbf{K}\mathbf{b} = \mathbf{b}\cdot\mathbf{K}\mathbf{a};\;\; \mathbf{a}\cdot\mathbf{M}\mathbf{b} = \mathbf{b}\cdot\mathbf{M}\mathbf{a} \quad \forall \mathbf{a},\mathbf{b}\in V,$$

$$\mathbf{a}\cdot\mathbf{K}\mathbf{a} \geq 0 \quad \forall \mathbf{a}\in V.$$

These relations imply the inequalities [1]

$$2\mathbf{A}\cdot\mathbf{C}[\mathbf{B}] \leq \xi^{-1}\mathbf{A}\cdot\mathbf{C}[A] + \xi\mathbf{B}\cdot\mathbf{C}[\mathbf{B}] \quad \forall \mathbf{A},\mathbf{B}\in Lin,\; \forall\xi>0 \tag{3.4}$$

$$2\mathbf{a}\cdot\mathbf{K}\mathbf{b} \leq \xi^{-1}\mathbf{a}\cdot\mathbf{K}\mathbf{a} + \xi\mathbf{b}\cdot\mathbf{K}\mathbf{b} \quad \forall \mathbf{a},\mathbf{b}\in V,\; \forall\xi>0. \tag{3.5}$$

We also assume that $\rho^{-1}(\mathbf{x})$, $c_e^{-1}(\mathbf{x})$, $c_\alpha^{-1}(\mathbf{x})$, $|\mathbf{C}(\mathbf{x})|$, $|\mathbf{K}(\mathbf{x})|$ and $|\mathbf{M}(\mathbf{x})|$ are bounded on $\bar{C}$. The elasticity tensor $\mathbf{C}$ maps Lin into the subspace Sym of all symmetric elements of Lin: its kernel is the whole subspace Skw of all skew-symmetric elements of Lin. Finally, we assume that the material fields have the following regularity properties:

$$\rho,\; c_e,\; c_\alpha,\; m,\; n,\; \in C^0(\bar{C}),\; \mathbf{K},\; \mathbf{M}\in C_2^1(\bar{C}),\;\; \mathbf{C}\in C_4^1(\bar{C}), \tag{3.6}$$

where, as usual, $\bar{C} = C\cup\partial C$. Moreover we suppose that the external fields $\mathbf{b}(\mathbf{x},t)$ and $r(\mathbf{x},t)$ are such that

$$\mathbf{b}\in C^1(Q),\;\; r\in C^1(\bar{Q}), \tag{3.7}$$

where $\bar{Q} = \bar{C}\times[0,+\infty[$. As far as the field equations are concerned, let us observe that equations (2.4) and (2.14) imply

$$c_e\dot{\hat{\theta}} + c_\alpha\dot{\hat{\alpha}} = \theta_0\mathbf{M}\cdot\nabla\dot{\hat{\mathbf{u}}} + \operatorname{div}(\mathbf{K}\nabla\theta) + \hat{\mathrm{r}}, \tag{3.8}$$

where

$$\hat{f} = f + t_0\dot{f} \tag{3.9}$$

and θ_0 is a suitable reference temperature. Hence, our system of equations becomes now

$$\rho\ddot{\mathbf{u}} = \operatorname{div}(\mathbf{C}[\mathbf{E}] + \theta\mathbf{M}) + \mathbf{b}, \tag{3.10}$$

$$c_e\dot{\hat{\theta}} + c_\alpha\dot{\hat{\alpha}} = \theta_0\mathbf{M}\cdot\nabla\dot{\hat{\mathbf{u}}} + \operatorname{div}(\mathbf{K}\nabla\theta) + \hat{\mathrm{r}}, \tag{3.11}$$

$$\dot{\alpha} = m\theta + n\alpha. \tag{3.12}$$

Let now $\{\partial_1 C, \partial_2 C\}$ and $\{\partial_3 C, \partial_4 C\}$ be two partitions of the boundary ∂C of C such that

$$\partial C = \partial_1 C\cup\partial_2 C = \partial_3 C\cup\partial_4 C,$$
$$\partial_1 C\cap\partial_2 C = \emptyset,\;\; \partial_3 C\cap\partial_4 C = \emptyset.$$

A solution of the mixed coupled thermoelasticity with thermal relaxation and an internal variable consists of a triple $\{\mathbf{u}, \theta, \alpha\}$ such that:

1. Equations (3.10)-(3.12) are satisfied;

2.

$$\mathbf{u} = \mathbf{u}_0, \ \dot{\mathbf{u}} = \dot{\mathbf{u}}_0,$$
$$\theta = \theta_0, \ \dot{\theta} = \dot{\theta}_0, \qquad \text{on } C \times \{0\}$$
$$\alpha = \alpha_0,$$

where $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, θ_0, $\dot{\theta}_0$ and α_0 are given initial conditions;

3.

$$\mathbf{u} = \mathbf{u}^* \ \text{ on } \partial_1 C \times [0, +\infty[,$$
$$\mathbf{S}\mathbf{n} \equiv (\mathbf{C}[\nabla \mathbf{u}] + \mathbf{M}\theta)\mathbf{n} = \mathbf{s}^* \ \text{ on } \partial_2 C \times [0, +\infty[,$$
$$\theta = \theta^* \ \text{ on } \partial_3 C \times [0, +\infty[,$$
$$-\mathbf{K}\nabla\theta \cdot \mathbf{n} = q^* \ \text{ on } \partial_4 C \times [0, +\infty[,$$

with $\mathbf{n}$ being the outward unit normal to ∂C.

In the next section we will prove a suitable inequality representing a link between the support of the data $\mathbf{u}^*$, $\mathbf{s}^*$, θ^*, q^*, $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, θ_0, $\dot{\theta}_0$, α_0, r and $\mathbf{b}$, and the support of the solution $(\mathbf{u}, \theta, \alpha)$ at each instant $t > 0$.

4. A domain of dependence inequality

The present section is devoted to prove a domain of dependence inequality for a solution $(\mathbf{u}, \theta, \alpha)$ of the initial and boundary value problem of Sec. 3.

Let us first observe that, if we multiply the equations (3.10) and (3.12) by t_0, then differentiate with respect to t and finally add the derived equations to the original ones, we may rewrite system (3.10)-(3.12) as follows:

$$\rho\ddot{\hat{\mathbf{u}}} = \operatorname{div}\{\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M}\} + \hat{\mathbf{b}}, \tag{4.1}$$

$$c_e\dot{\hat{\theta}} + c_\alpha\dot{\hat{\alpha}} = \theta_0\mathbf{M}\cdot\nabla\dot{\hat{\mathbf{u}}} + \operatorname{div}(\mathbf{K}\nabla\theta) + \hat{\mathrm{r}}, \tag{4.2}$$

$$\dot{\hat{\alpha}} = m\hat{\theta} + n\,\hat{\alpha}. \tag{4.3}$$

For this system we prove the following

THEOREM 1. *(Domain of dependence inequality). Let* $(\mathbf{u}, \theta, \alpha)$ *be a solution of the initial and boundary value problem specified in Sec. 3, and let c be a positive constant of the velocity dimension, such that*

$$c^{-1}|\mathbf{A}| + \{(c_e \rho^{-1})\theta_0\}^{\frac{1}{2}}|\mathbf{M}| \le c, \tag{4.4}$$

$$(c\, c_e\, t_0)^{-1}|\mathbf{K}| + \{(c_e \rho)^{-1}\theta_0\}^{\frac{1}{2}}|\mathbf{M}| \le c, \tag{4.5}$$

where $\mathbf{A}$ *is the"acoustic tensor" in the direction of propagation* $\mathbf{m}$*, defined by*

$$\mathbf{A}(\mathbf{x},\mathbf{m}) = \rho^{-1}(\mathbf{x})\mathbf{C}[\mathbf{a} \otimes \mathbf{m}] \;\; \forall \mathbf{x} \in E_3, \;\; \forall \mathbf{a} \in V; \tag{4.6}$$

then the following domain of dependence inequality holds true:

$$\begin{aligned}
&\int\limits_{C(\mathbf{x}_0,R)} \eta(\mathbf{x},t)dv + \theta_0^{-1}\int\limits_0^t ds \int\limits_{C[\mathbf{x}_0,R+c(t-s)]} (\nabla\theta \cdot \mathbf{K}\nabla\theta)(\mathbf{x},s)dv \\
&\quad + \theta_0^{-1}\int\limits_0^t ds \int\limits_{C[\mathbf{x}_0,R+c(t-s)]} c_\alpha \frac{1}{m}\dot{\hat{\alpha}}^2(\mathbf{x},s)dv \le \int\limits_{C[\mathbf{x}_0,R+ct]} \eta(\mathbf{x}_0,0)dv \\
&\quad + \int\limits_0^t ds \int\limits_{C[\mathbf{x}_0,R+c(t-s)]} (\hat{\mathbf{b}} \cdot \hat{\mathbf{u}} + \theta_0^{-1}\hat{r}\hat{\theta})(\mathbf{x},s)dv \\
&\quad + \int\limits_0^t ds \int\limits_{\partial C \cap S[\mathbf{x}_0,R+c(t-s)]} (\dot{\hat{\mathbf{u}}} \cdot \hat{\mathbf{s}} - \theta_0^{-1}\hat{\theta}\hat{q})(\mathbf{x},s)d\sigma
\end{aligned} \tag{4.7}$$

$\forall t > 0$, $\forall R > 0$ and $\forall \mathbf{x}_0 \in C$,
where

$$\begin{aligned}
&\eta(\mathbf{x},s) \\
&= \frac{1}{2}\Big(\rho\dot{\hat{\mathbf{u}}}^2 + \nabla\hat{\mathbf{u}} \cdot \mathbf{C}[\nabla\hat{\mathbf{u}}] + c_e\theta_0^{-1}\hat{\theta}^2 + t_0\theta_0^{-1}\nabla\theta \cdot \mathbf{K}\nabla\theta - c_\alpha\theta_0^{-1}\frac{n}{m}\hat{\alpha}^2\Big)(\mathbf{x},s)
\end{aligned} \tag{4.8}$$

while

$$S(\mathbf{x}_0,d) = \{\mathbf{x} \in E_3 : |\mathbf{x} - \mathbf{x}_0| < d\},$$

$$C(\mathbf{x}_0,d) = C \cap S(\mathbf{x}_0,d)$$

$\forall d \in R^+$ *and, moreover,*

$$(\mathbf{C}[\nabla\hat{\mathbf{u}}] + \mathbf{M}\hat{\theta})\,\mathbf{n} = \hat{\mathbf{s}}, \tag{4.9}$$

$$-(\mathbf{K}\nabla\hat{\theta})\cdot\mathbf{n} = \hat{q}. \tag{4.10}$$

Proof. Let $g_\delta : \lambda \in R \to g_\delta(\lambda) \in [0,1]$ such that

$$\begin{gathered} g_\delta(\lambda) = 0 \;\; if \;\; \lambda \in]-\infty, 0], \\ g_\delta(\lambda) = 1 \;\; if \;\; \lambda \in [\delta, +\infty[, \;\; \delta \geq 0, \\ g'_\delta(\lambda) = \frac{dg_\delta(\lambda)}{d\lambda} \geq 0 \;\; \forall\lambda \in R, \end{gathered} \tag{4.11}$$

and let us define

$$g(\mathbf{x}, s) \equiv g_\delta(c^{-1}[R + c(t-s) - |\mathbf{x} - \mathbf{x}_0|]), \tag{4.12}$$

where R is a positive constant of the length dimension while t and $\mathbf{x}_0$ are arbitrarily fixed. Function g is defined on $E_3 \times [0, +\infty[$ and its support is

$$\Sigma = \bigcup_{s\in[0,t]} S[\mathbf{x}_0, R + c(t-s)]. \tag{4.13}$$

Moreover, g is smooth on $E_3 \times [0, +\infty[$ and ∇g vanishes identically on the following set

$$\Sigma_0 = \bigcup_{s\in[0,t]} S[\mathbf{x}_0, R + c(t-s-\delta)] \tag{4.14}$$

in which

$$g(\mathbf{x}, s) = 1 \;\; \forall\mathbf{x} \in S[\mathbf{x}_0, R + c(t-s-\delta)]. \tag{4.15}$$

Clearly, δ is to be choosen so small as to assure that $R + c(t-s-\delta) > 0$ for any $s \in [0,t]$. Now if we multiply both sides of (4. 1) by $g\dot{\hat{\mathbf{u}}}$ and use the vectorial identity relative to the divergence of an inner product, we obtain

$$\begin{aligned} \frac{1}{2}\, g\, \frac{d}{dt}\left[\rho\dot{\hat{\mathbf{u}}}\cdot\dot{\hat{\mathbf{u}}}\right] &= g\hat{\mathbf{b}}\cdot\dot{\hat{\mathbf{u}}} + \nabla\cdot\{(\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M})\,g\dot{\hat{\mathbf{u}}}\} \\ &\quad - g\{\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M}\}\cdot\nabla\dot{\hat{\mathbf{u}}} - \dot{\hat{\mathbf{u}}}\cdot\{\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M}\}\nabla g. \end{aligned} \tag{4.16}$$

As a further step, we multiply (4.2) by $\hat{\theta}$ and take into account the definition of $\hat{\theta}$ in order to get

$$\hat{\theta}\nabla\dot{\hat{\mathbf{u}}}\cdot\mathbf{M} - c_\alpha\dot{\hat{\alpha}}\hat{\theta}\theta_0^{-1} = (2\theta_0)^{-1}\frac{d}{dt}(c_e\hat{\theta}^2) - \theta_0^{-1}\nabla\cdot(\hat{\theta}\mathbf{K}\nabla\theta) + \theta_0^{-1}\nabla\theta\cdot\mathbf{K}\nabla\theta + t_0(2\theta_0)^{-1}\frac{d}{dt}(\nabla\theta\cdot\mathbf{K}\nabla\theta) - \hat{r}\hat{\theta}\theta_0^{-1}. \tag{4.17}$$

Finally, we substitute equation (4.17) in (4.16) and integrate on $C\times[0,t]$. Then, by applying the divergence theorem and taking into account (4.9) and (4.10), we may write

$$\begin{aligned}
&\int\limits_C \Big[\frac{1}{2}g\rho\dot{\hat{\mathbf{u}}}^2\Big]_0^t dv + \int\limits_C \Big[\frac{1}{2}g\,c_e\theta_0^{-1}(\hat{\theta})^2\Big]_0^t dv + \int\limits_C \Big[\frac{1}{2}gt_0\theta_0^{-1}(\nabla\theta\cdot\mathbf{K}\nabla\theta)\Big]_0^t dv \\
&\quad + \int\limits_C \Big[\frac{1}{2}g\,\nabla\hat{\mathbf{u}}\cdot\mathbf{C}[\nabla\hat{\mathbf{u}}]\Big]_0^t dv + \theta_0^{-1}\int\limits_0^t ds\int\limits_C (g\nabla\theta\cdot\mathbf{K}\nabla\theta)(\mathbf{x},s)dv, \\
&= \int\limits_0^t ds\int\limits_C \Big[\frac{1}{2}\rho\dot{\hat{\mathbf{u}}}^2\dot{g} + \frac{1}{2}\theta_0^{-1}c_e\hat{\theta}^2\dot{g} + \frac{1}{2}t_0\theta_0^{-1}\nabla\theta\cdot\mathbf{K}\nabla\theta\dot{g} + \frac{1}{2}\nabla\hat{\mathbf{u}}\cdot\mathbf{C}[\nabla\hat{\mathbf{u}}]\dot{g}\Big]dv + \\
&\quad - \int\limits_0^t ds\int\limits_C \dot{\hat{\mathbf{u}}}\cdot(\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M})\nabla g dv - \int\limits_0^t ds\int\limits_C \theta_0^{-1}\hat{\theta}\nabla g\cdot\mathbf{K}\nabla\theta dv \\
&\quad + \int\limits_0^t ds\int\limits_{\partial C} g[\dot{\hat{\mathbf{u}}}\cdot\hat{\mathbf{s}} - \theta_0^{-1}\hat{\theta}\hat{q}]d\sigma + \int\limits_0^t ds\int\limits_C [g(\hat{\mathbf{b}}\cdot\dot{\hat{\mathbf{u}}} + \hat{r}\hat{\theta}\theta_0^{-1})]\,dv + \\
&\quad - \int\limits_0^t ds\int\limits_C g(c_\alpha\dot{\hat{\alpha}}\hat{\theta}\theta_0^{-1})dv.
\end{aligned} \tag{4.18}$$

On the other hand, from (4.3) it follows that

$$\hat{\theta}\dot{\hat{\alpha}} = \tau\dot{\hat{\alpha}}^2 - \sigma\hat{\alpha}\dot{\hat{\alpha}} \tag{4.19}$$

with

$$\tau \equiv \frac{1}{m}, \quad \sigma \equiv \frac{n}{m}. \tag{4.20}$$

Then, due to (4.19), the last integral of (4.18) takes the form

$$\int_0^t ds \int_C g c_\alpha \theta_0^{-1} \dot{\hat{\alpha}}\hat{\theta}\, dv = \int_0^t ds \int_C g c_\alpha \theta_0^{-1} \tau \dot{\hat{\alpha}}^2\, dv + \\ - \frac{1}{2}\int_C [g c_\alpha \theta_0^{-1}\sigma\hat{\alpha}^2]_0^t dv + \frac{1}{2}\int_0^t ds \int_C c_\alpha \theta_0^{-1}\sigma\hat{\alpha}^2 \dot{g}\, dv. \tag{4.21}$$

Owing to the definition of η and taking into account the equalities presented above, the following relation holds true:

$$\int_C g\eta(\mathbf{x},t)\, dv + \theta_0^{-1}\int_0^t ds \int_C (g\nabla\theta \cdot \mathbf{K}\nabla\theta)(\mathbf{x},s) dv \\ = \int_C g\eta(\mathbf{x},0)\, dv + \int_0^t ds \int_C (\dot{g}\eta)(\mathbf{x},s)\, dv - \int_0^t ds \int_C \dot{\hat{\mathbf{u}}} \cdot \{\mathbf{C}[\nabla\hat{\mathbf{u}}] + \hat{\theta}\mathbf{M}\}\nabla g\, dv + \\ - \int_0^t ds \int_C \theta_0^{-1}\hat{\theta}\nabla g \cdot \mathbf{K}\nabla\theta\, dv + \int_0^t ds \int_{\partial C} g(\hat{\mathbf{u}} \cdot \hat{\mathbf{s}} - \theta_0^{-1}\hat{\theta}\hat{q}) d\sigma \\ + \int_0^t ds \int_C g(\hat{\mathbf{b}} \cdot \dot{\hat{\mathbf{u}}} + \theta_0^{-1}\hat{r}\hat{\theta})\, dv - \int_0^t ds \int_C g c_\alpha \theta_0^{-1}\tau\dot{\hat{\alpha}}^2 dv. \tag{4.22}$$

The third and the fourth integrals at the right-hand side of equation (4.22) may be estimated by using the following inequality, which follows from (3.4), (3.5), (4.4), (4.5) together with the definition of $g(\mathbf{x}, s)$ and $\mathbf{A}(\mathbf{x}, \mathbf{m})$:

$$(4.23)\quad -\dot{\hat{\mathbf{u}}}\cdot\mathbf{C}[\nabla\hat{\mathbf{u}}]\nabla g-\dot{\hat{\mathbf{u}}}\cdot\mathbf{M}\nabla g-\theta_0^{-1}\hat{\theta}\nabla g\cdot(\mathbf{K}\nabla\theta)$$

$$\leq\frac{1}{2}g_\delta'\Big\{\nabla\hat{\mathbf{u}}\cdot\mathbf{C}[\nabla\hat{\mathbf{u}}]+(c^{-2}\dot{\hat{\mathbf{u}}}\cdot\mathbf{C}[\dot{\hat{\mathbf{u}}}\otimes\mathbf{e}_\mathbf{r}^0]\mathbf{e}_\mathbf{r}^0)\Big\}$$

$$+\frac{1}{2}g_\delta'\Big\{c^{-1}|\mathbf{M}|[(c_e\rho)^{-1}\theta_0]^{\frac{1}{2}}(\rho\dot{\hat{\mathbf{u}}}^2+\theta_0^{-1}c_e\hat{\theta}^2)$$

$$+(c^2t_0\theta_0)^{-1}\hat{\theta}^2\mathbf{e}_r^0\cdot\mathbf{K}\mathbf{e}_r^0+\theta_0^{-1}t_0\nabla\theta\cdot\mathbf{K}\nabla\theta\Big\}$$

$$\leq\frac{1}{2}g_\delta'\Big\{\nabla\hat{\mathbf{u}}\cdot\mathbf{C}[\nabla\hat{\mathbf{u}}]+(c^{-2}|\mathbf{A}|+c^{-1}|\mathbf{M}|\Big\{(c_e\rho)^{-1}\theta_0\Big\}^{\frac{1}{2}})\rho\dot{\hat{\mathbf{u}}}^2$$

$$+[(c^2t_0c_e)^{-1}|\mathbf{K}|+c^{-1}|\mathbf{M}|\Big\{(c_e\rho)^{-1}\theta_0\Big\}^{\frac{1}{2}}]c_e\theta_0^{-1}\hat{\theta}^2+\theta_0^{-1}t_0\nabla\theta\cdot\mathbf{K}\nabla\theta\Big\}\leq g_\delta'\eta.$$

Here

$$\mathbf{e}_r^0=|\mathbf{x}-\mathbf{x}_0|^{-1}(\mathbf{x}-\mathbf{x}_0).$$

Let us observe now that function g is not decreasing and hence the relation

$$(4.24)\qquad g_\delta'(\lambda)=-\dot{g}(\mathbf{x},s)$$

implies that $\dot{g}$ is negative. As a consequence, owing to (4.23), the sum of the second, third and fourth integrals on the right-hand side of (4.22) is negative, so that from (4.22) it follows:

$$(4.25)\quad \int_C g\eta(\mathbf{x},t)dv+\theta_0^{-1}\int_0^t ds\int_C(g\nabla\theta\cdot\mathbf{K}\nabla\theta)(\mathbf{x},s)dv$$

$$+\theta_0^{-1}\int_0^t ds\int_C gc_\alpha\tau\dot{\hat{\alpha}}^2(\mathbf{x},s)dv\leq\int_C g\eta(\mathbf{x},0)dv$$

$$+\int_0^t ds\int_{\partial C}g(\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\cdot\mathbf{s}-\theta_0\hat{\theta}\hat{q})d\sigma+\int_0^t ds\int_C(\hat{\mathbf{b}}\cdot\dot{\hat{\mathbf{u}}}+\theta_0^{-1}\hat{r}\hat{\theta})dv.$$

Finally, since

$$g(\mathbf{x},s)=g_\delta(c^{-1}[R+c(t-s)-|\mathbf{x}-\mathbf{x}_0|]),$$

$$g_\delta(\lambda)=1\quad\forall\lambda\in[\delta,+\infty[$$

and g having support Σ defined by (4.13), we may conclude that, as δ tends to zero, function g tends to the characteristic function of the set Σ. Then, it is allowed to calculate the limit of (4.25) as δ tends to 0. It yields exactly (4.7), q.e.d.

5. Domain of influence theorem

In this section we prove that from the domain of dependence inequality, together with the hypotheses of Sec. 2 and 3, a domain of influence theorem follows. Let $\tilde{C}(t)$, $t > 0$, be the set of the points $\mathbf{x} \in \bar{C}$ such that:

1. $\mathbf{x} \in C \Rightarrow \mathbf{u}_0 \neq 0 \quad \text{or} \quad \dot{\mathbf{u}}_0 \neq 0, \quad \text{or} \quad \theta_0 \neq 0 \quad \text{or} \quad \dot{\theta}_0 \neq 0 \quad \text{or} \quad \alpha_0 \neq 0;$

   moreover $\exists \bar{t} \in [0,t] : \mathbf{b}(\mathbf{x},t) \neq 0 \quad \text{or} \quad r(\mathbf{x},\bar{t}) \neq 0;$

2. $\mathbf{x} \in \partial_1 C \Rightarrow \exists \bar{t} \in [0,t] : \mathbf{u}^*(\mathbf{x},\bar{t}) \neq \mathbf{0};$

3. $\mathbf{x} \in \partial_2 C \Rightarrow \exists \bar{t} \in [0,t] : \mathbf{s}^*(\mathbf{x},\bar{t}) \neq \mathbf{0};$

4. $\mathbf{x} \in \partial_3 C \Rightarrow \exists \bar{t} \in [0,t] : \theta^*(\mathbf{x},\bar{t}) \neq 0;$

5. $\mathbf{x} \in \partial_4 C \Rightarrow \exists \bar{t} \in [0,t] : q^*(\mathbf{x},\bar{t}) \neq 0.$

The set

$$C^* = \{\mathbf{x}_0 \in \bar{C} : \tilde{C}(t) \cap \bar{S}(\mathbf{x}_0, ct) \neq \emptyset\}, \tag{5.1}$$

where c is the same of equations (4.4)-(4.5), is the *domain of influence* of the data at the instant t. We prove now the following

THEOREM 2. *(Domain of influence theorem). Let* $(\mathbf{u}, \theta, \alpha)$ *be a solution of the initial and boundary value problem (3.10)-(3.12). Then* $\mathbf{u} = \mathbf{0}$, $\theta = 0$ *and* $\alpha = 0$ *on the set* $\bar{Q} = \{\bar{C} - C^*\} \times [0,t]$.

Proof. Let $\mathbf{x}_0 \in \bar{C} - C^*(t)$ and let $\lambda \in [0,t]$. We apply the domain of dependence inequality with $t = \lambda$ and $R = c(t-\lambda)$ in order to get

$$(5.2)\qquad \int_{C[\mathbf{x}_0,c(t-\lambda)]} \eta(\mathbf{x},\lambda)dv + \theta_0^{-1}\int_0^{\lambda} ds \int_{C[\mathbf{x}_0,c(t-s)]} (\nabla\theta\cdot\mathbf{K}\nabla\theta)(\mathbf{x},s)dv$$
$$+\theta_0^{-1}\int_0^{\lambda} ds \int_{C[\mathbf{x}_0,c(t-s)]} (c_\alpha\tau\dot{\hat{\alpha}}^2)(\mathbf{x},s)\mathrm{dv} \le \int_{\mathrm{C}(\mathbf{x}_0,\mathrm{ct})} \eta(\mathbf{x},0)\mathrm{dv}$$
$$+\int_0^{\lambda} ds \int_{C[\mathbf{x}_0,c(t-s)]} (\hat{\mathbf{b}}\cdot\dot{\hat{\mathbf{u}}} + \hat{\theta}_0^{-1}\hat{r}\hat{\theta})(\mathbf{x},s)dv$$
$$+\int_0^{\lambda} ds \int_{\partial C\cap S[\mathbf{x}_0,c(t-s)]} (\dot{\hat{\mathbf{u}}}\cdot\hat{\mathbf{s}} - \theta_0^{-1}\hat{\theta}\hat{q})(\mathbf{x},s)d\sigma.$$

The right hand-side of (5.2) is zero. In fact, because $\mathbf{x}_0 \in (\bar{C} - B^*(t))$, by equation (3.10) restricted to the set $C(\mathbf{x}_0, ct)\times\{0\}$ and by Hypothesis 1, it follows $\ddot{\mathbf{u}}_0 = \mathbf{0}$ and hence $\dot{\hat{\mathbf{u}}} = \mathbf{0}$ on $C(\mathbf{x}_0, ct)$. Also by Hypothesis 1 we have $\hat{\theta}_0 = 0$, $\nabla\hat{\mathbf{u}} = \mathbf{0}$ and $\nabla\theta_0 = \mathbf{0}$ on $C(\mathbf{x}_0, ct)$. This is enough to conclude that

$$(5.3)\qquad \int_{C(\mathbf{x}_0,ct)} \eta(\mathbf{x},0)dv = 0.$$

Furthermore $r(\mathbf{x},s)$ and $\mathbf{b}(\mathbf{x},s)$ vanish on $C(\mathbf{x}_0,ct)\times[0,t]$ so that $\dot{\mathbf{b}}(\mathbf{x},s)$ and $\dot{r}(\mathbf{x},s)$ are zero on the same set. As a consequence, $\hat{\mathbf{b}}(\mathbf{x},s)$ and $\hat{r}(\mathbf{x},s)$ identically vanish on $C(\mathbf{x}_0,ct)\times[0,t]$, so that

$$(5.4)\qquad \int_0^{\lambda} ds \int_{C[\mathbf{x}_0,c(t-s)]} (\hat{\mathbf{b}}\cdot\dot{\hat{\mathbf{u}}} + \hat{\theta}_0^{-1}\hat{r}\hat{\theta})(\mathbf{x},s)dv = 0.$$

In order to evaluate the last integral in (5.2) we rewrite it in the form

$$(5.5)\quad \int_0^\lambda ds \int_{\partial C\cap S[\mathbf{x}_0,c(t-s)]} (\dot{\hat{\mathbf{u}}}\cdot\hat{\mathbf{s}} - \theta_0^{-1}\hat\theta\hat q)(\mathbf{x},s)d\sigma$$

$$= \int_0^\lambda \Bigg[\int_{\partial_1 C\cap S[\mathbf{x}_0,c(t-s)]} (\dot{\mathbf{u}} + t_0\ddot{\mathbf{u}})\cdot\hat{\mathbf{s}}d\sigma + \int_{\partial_2 C\cap S[\mathbf{x}_0,c(t-s)]} \dot{\hat{\mathbf{u}}}\cdot(\mathbf{s}_0 + t_0\dot{\mathbf{s}})d\sigma +$$

$$- \theta_0^{-1} \int_{\partial_3 C\cap S[\mathbf{x}_0,c(t-s)]} (\theta + t_0\dot\theta)\hat q d\sigma - \theta_0^{-1} \int_{\partial_4 C\cap S[\mathbf{x}_0,c(t-s)]} \hat\theta\hat q d\sigma\Bigg] ds.$$

Since $\lambda \le t$ and $C(\mathbf{x}_0, ct) \subset \{\bar C - \tilde C(t)\}$, due to Conditions 2-5, the right-hand side of (5.5) also vanishes. Then, inequality (5.2) reduces to

$$(5.6)\quad \int_C \eta(\mathbf{x},\lambda)dv + \theta_0^{-1}\int_0^\lambda ds \int_{C[\mathbf{x}_0,c(t-\lambda)]} (\nabla\theta\cdot\mathbf{K}\nabla\theta)(\mathbf{x},s)dv+$$

$$+ \theta_0^{-1}\int_0^\lambda ds \int_{C[\mathbf{x}_0,c(t-\lambda)]} c_\alpha\tau\dot{\hat\alpha}^2(\mathbf{x},s)dv \le 0.$$

Let us recall now that tensor $\mathbf{K}$ is positive semidefinite and from (5.6) it follows that

$$(5.7)\quad \int_{C[\mathbf{x}_0,c(t-\lambda)]} \eta(\mathbf{x},\lambda)dv \le 0.$$

Finally, since η is non-negative, we have

$$(5.8)\quad \eta(\mathbf{x},\lambda) = 0.$$

Taking into account the definition of η we deduce

$$(5.9)\quad \dot{\hat{\mathbf{u}}}(\mathbf{x}_0,\lambda) = \mathbf{0},\quad \hat\theta(\mathbf{x}_0,\lambda) = 0,\quad \text{and}\quad \hat\alpha(\mathbf{x}_0,\lambda) = 0.$$

Moreover, Condition 1 implies

$$(5.10)\quad \hat{\mathbf{u}}(\mathbf{x}_0,0) = \mathbf{0},\quad \theta(\mathbf{x}_0,0) = 0,\quad \alpha(\mathbf{x}_0,0) = 0\quad \forall\mathbf{x}_0 \in \bar C - C^*(t).$$

Hence, from the uniqueness theorem of the solution of ordinary differential equations it follows

$$(5.11)\quad \hat{\mathbf{u}}(\mathbf{x}_0,\lambda) = \mathbf{0},\ \theta(\mathbf{x}_0,\lambda) = 0,\ \alpha(\mathbf{x}_0,\lambda) = 0\ \forall(\mathbf{x}_0,\lambda) \in \{\bar{C} - C^*(t)\} \times [0,t].$$

Finally, the ordinary differential equation

$$(5.12)\qquad \hat{\mathbf{u}}(\mathbf{x}_0,\lambda) = \mathbf{0},$$

with the initial condition

$$(5.13)\qquad \mathbf{u}(\mathbf{x}_0,0) = \mathbf{0},$$

admits the unique solution

$$(5.14)\qquad \mathbf{u}(\mathbf{x}_0,\lambda) = \mathbf{0}\ \forall\lambda \in [0,t].$$

This is enough to conclude that

$$(5.15)\qquad \mathbf{u}(\mathbf{x},\lambda) = \mathbf{0}\ \forall\,(\mathbf{x},\lambda) \in \{\bar{C} - C^*(t)\} \times [0,t].$$

The theorem has been proved.

6. The semi-empirical heat conduction model

One of the most fundamental and delicate concepts of non-equilibrium thermodynamics is that of absolute temperature. Its definition is well founded at the equilibrium or even for small deviations from an equilibrium state. However, it becomes questionable for arbitrary states and processes when the Clausius integral extended to a closed process is not zero [23]. On the other hand, it becomes necessary to consider non-equilibrium states and processes if one wants to describe some important thermodynamic phenomena. One of these is just the propagation of thermal waves at a low temperature. Kosiński and co-workers approached the problem by introducing a new temperature β, called semi-empirical, as a scalar internal state variable, [10, 12]. It is related to the absolute temperature, θ, by a suitable ordinary differential equation of the type

$$(6.1)\qquad \dot{\beta} = f(\theta, t)$$

and a given initial condition

$$(6.2)\qquad \beta(t_0) = \beta_0.$$

Obviously, the new theory is hyperbolic but it must also allow a passage to the classical parabolic case given by Fourier's law. By design, when relaxed β

coincides with the absolute temperature θ, otherwise β follows after θ with a certain delay, controlled by a small parameter τ, called relaxation time. This delay introduces hyperbolicity and, if chosen to be small, controls the passage to the classical case. A general model of anisotropic thermoelastic solids with semi-empirical temperature has been introduced in [12]. In the linear case, numerical solutions to an initial and boundary value problem of the type considered in the present paper have been found in [19]. These solutions confirm that the model admits hyperbolic heat propagation. In this section we apply the result of Sec. 5 in order to prove the hyperbolicity of the theory via a domain of influence theorem. To this end let us consider a linear thermoelastic solid described by a set of constitutive equations having the form

$$\Phi = \Phi^*(\theta, \beta, \nabla\theta, \nabla\beta, \mathbf{E}). \tag{6.3}$$

The additional hypotheses quoted below specify better the model.

a) The stress tensor depends on β only through θ and is given by the constitutive equation

$$\mathbf{S}(\mathbf{E}, \theta) = \mathbf{C}[\nabla\mathbf{u}] + \mathbf{M}\theta, \tag{6.4}$$

where $\mathbf{C}$ and $\mathbf{M}$ are the same of Sec. 2.

b) The evolution of the semi-empirical temperature is determined by the linear differential equation

$$\dot{\beta} = \frac{1}{\tau}\theta - \frac{1}{\sigma}\beta, \tag{6.5}$$

where the material scalar parameters $\tau = \tau(x)$ and $\sigma = \sigma(x)$ are positive and have both the dimensions of time.

c) The heat flux is given by the Fourier's type heat conduction law

$$\mathbf{q} = -\mathbf{K}\nabla\beta, \tag{6.6}$$

where $\mathbf{K}(x)$ is the the heat conductivity tensor defined in Sec. 2.

d) The specific internal energy ϵ does not depend on $\nabla\beta$ [2)], i.e.

[2)]Let us recall that the second law of thermodynamics prevents ϵ from depending on $\nabla\theta$.

$$\epsilon = \epsilon(\theta, \beta, \mathbf{E}). \tag{6.7}$$

Equation (6.5) may be rewritten as follows

$$\tau\dot{\beta} = \theta - \frac{\tau}{\sigma}\beta. \tag{6.8}$$

Taking the gradient of (6.8) and applying to the obtained equation the operator $-\mathbf{K}$, we get

$$\sigma\dot{\mathbf{q}} + \mathbf{q} = -\tilde{\mathbf{K}}\nabla\theta, \tag{6.9}$$

with $\tilde{\mathbf{K}} = \dfrac{\sigma}{\tau}\mathbf{K}$. Equation (6.9) is of the Cattaneo's type and it may be put in the form

$$\hat{\mathbf{q}} = -\tilde{\mathbf{K}}\nabla\theta, \tag{6.10}$$

with

$$\hat{f} = f + \sigma\dot{f}. \tag{6.11}$$

The same procedure developed in Sec. 3 allows us to write our system of equations in the form

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\{\mathbf{C}[\nabla\mathbf{u}] + \theta\mathbf{M}\} + \mathbf{b}, \tag{6.12}$$

$$c_e\dot{\hat{\theta}} + c_\beta\dot{\beta} = \theta_0\mathbf{M}\cdot\nabla\dot{\hat{\mathbf{u}}} + \operatorname{div}\{\tilde{\mathbf{K}}\nabla\theta\} + \hat{r}, \tag{6.13}$$

$$\dot{\beta} = \frac{1}{\tau}\theta - \frac{1}{\sigma}\beta, \tag{6.14}$$

where c_β is the latent heat relative to the semi-empirical temperature β. We assume for the functions $\dfrac{1}{\tau}$, $-\dfrac{1}{\sigma}$ and c_β the same properties of m, n and c_α, respectively. Then, we consider the problem of finding a solution $(\mathbf{u}, \theta, \beta)$ of equations (6.12)-(6.14) such that

$$\mathbf{u} = \mathbf{u}_0, \ \ \dot{\mathbf{u}} = \dot{\mathbf{u}}_0, \ \ \theta = \theta_0, \ \ \dot{\theta} = \dot{\theta}_0 \ \ \beta = \beta_0 \ \text{ on } \ C\times\{0\}, \tag{6.15}$$

where $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, θ_0, $\dot{\theta}_0$ and β_0 are given initial functions, and

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^* \ \text{ on } \ \partial_1 C \times [0, +\infty[, \\ (\mathbf{C}[\nabla\mathbf{u}] + \mathbf{M}\theta)\mathbf{n} &= \mathbf{s}^* \ \text{ on } \ \partial_2 C \times [0, +\infty[, \\ \theta &= \theta^* \ \text{ on } \ \partial_3 C \times [0, +\infty[, \\ -\mathbf{K}\nabla\theta\cdot\mathbf{n} &= q^* \ \text{ on } \ \partial_4 C \times [0, +\infty[. \end{aligned} \tag{6.16}$$

It is immediately seen that, by identifying β with α, the above mentioned initial and boundary value problem is the same as that of Sec. 3. Then, we define the domain of influence C^* of the data in the same manner as in Sec. 5. Theorem 4. admits the following corollary:

THEOREM 3. *Let* $(\mathbf{u}, \theta, \beta)$ *be a solution of the initial and boundary value problem (6.12)-(6.16). Then*

$$\mathbf{u} = 0, \;\; \theta = 0 \text{ and } \beta = 0 \tag{6.17}$$

on the domain

$$\tilde{Q} = \{\bar{C} - C^*\} \times [0, t]. \tag{6.18}$$

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