

On the macroscopic modelling of elastic/viscoplastic composites

CZ. WOŹNIAK and E. WIERZBICKI

*Institute of Mathematics and Informatics
Technological University of Częstochowa,
Dąbrowskiego 73, 42-200 Częstochowa, Poland
e-mail: wozniak@matinf.pcz.czyst.pl*

*Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday*

THE AIM of this contribution is to formulate a macroscopic model for the analysis of dynamic problems in micro-periodic composites made of elastic/viscoplastic and/or linear viscoelastic materials. The proposed modelling approach is based on the concept of tolerance averaging which so far was applied to the linear elastodynamics and heat transfer in periodic materials and structures. The obtained model equations, in contrast to homogenized equations, describe the effect of microstructure size on the overall behaviour of a composite solid.

1. Introduction

THIS PAPER is devoted to macroscopic modelling of certain inelastic micro-periodic composites. This modelling problem has been investigated for viscoelastic and elastic-plastic materials in a series of papers [2,4,5,10-15]. In this contribution we propose a unified method of macroscopic modelling for dynamic problems in micro-periodic composites made of elastic/viscoplastic and/or linear viscoelastic components. The motivation for writing this paper is an important role which recently play elastic/viscoplastic materials both from the theoretical and engineering point of view; among the leading papers on this subject we have to mention those by PERZYNA [6-9], to whom this work is dedicated. An alternative approach to the concept of elastic/viscoplastic materials can be found in [1] (cf. also [12] for the discussion of different models of viscoplasticity). In contrast to macroscopic models derived by homogenization, we look for models that make it possible to describe the effect of microstructure size on the overall dynamic behaviour of a micro-periodic solid. To this end we extend the tolerance averaging technique which so far has been applied to the problems of elastodynamics and heat conduction, [17,18,19]. In order to make the paper self-consistent, in the subsequent section, following [17], we outline some basic concepts of the tolerance averaging.

Throughout the paper we use the absolute tensor notation; by small bold letters, \mathbf{a} , \mathbf{b} , \mathbf{v} , \mathbf{w} ,... we denote vectors and vector fields and by capital boldface letters \mathbf{S} , \mathbf{D} ,... we denote the second order tensors and tensor fields. The block letters \mathbb{A} , \mathbb{B} , \mathbb{C} are reserved for the fourth order tensors and tensor fields. Symbol $\text{sym}(\mathbf{a} \otimes \mathbf{b})$ stands for a symmetric part of the second order tensor $\mathbf{a} \otimes \mathbf{b}$ and $\varepsilon(\mathbf{v})$ is a symmetrized gradient of an arbitrary differentiable vector field \mathbf{v} . Superscripts a, b, \dots and A, B, \dots run over sequences $1, \dots, n$ and $1, \dots, N$, respectively; summation convention holds unless otherwise stated. It is assumed that all introduced functions satisfy the smoothness conditions required in subsequent considerations.

2. Preliminaries

In this section we recall some basic concepts and statements related to the tolerance averaging, which will be used subsequently; for a detailed discussion the reader is referred to [17] (see also [18,19]). We begin with the statement that in the problem under consideration, every physical quantity (measured in a fixed system of units) can be specified only to within a certain tolerance. It means that the values F_1, F_2 of this quantity will be not discerned provided that $|F_1 - F_2| \leq \varepsilon_F$, where ε_F is a certain positive constant which is referred to as a tolerance parameter related to this quantity (cf. also [3], where ε_F is called "upper bound for negligibles"). In this case we shall tacitly assume that ε_F is known and we shall write $F_1 \cong F_2$. Hence \cong is a certain tolerance relation, i.e. a binary relation defined on \mathbb{R} which is symmetric and reflexive but not transitive. By a tolerance system we shall mean a mapping T which assigns to every unknown field F in the problem under consideration a tolerance parameter ε_F . Subsequently, we shall deal with fields which for every time t are defined in the region Ω in E^3 occupied by the composite. Let $\Delta = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ be a unit cell of the periodic structure of the composite and let $l = \text{diam}\Delta$. Let us denote by $\|\mathbf{x} - \mathbf{y}\|$ the distance between points \mathbf{x} and \mathbf{y} in E^3 , and by $B(\mathbf{x}, l)$ - the ball in E^3 with a center \mathbf{x} and a radius l . Setting $\Delta(\mathbf{x}) = \mathbf{x} + \Delta$, $\Omega_\Delta = \{\mathbf{x} \in \Omega : \Delta(\mathbf{x}) \subset \Omega\}$ we define the averaging operator of an arbitrary integrable function $f : \Omega \rightarrow \mathbb{R}$ by means of

$$\langle f \rangle(\mathbf{x}) = \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f(\mathbf{y}) dy_1 dy_2 dy_3, \quad \mathbf{x} \in \Omega_\Delta,$$

where $|\Delta|$ is the measure of Δ . If $f(\cdot)$ is a Δ -periodic function then $\langle f \rangle = \text{const}$. If $f(\cdot)$ depends also on time t then we shall write $\langle f \rangle(\mathbf{x}, t)$. Let DF stand for a function F as well as for all its derivatives (including time derivatives) which occur in the problem under consideration.

Function $F : \Omega \rightarrow \mathbb{R}$ will be called *slowly-varying*, $F \in SV_{\Delta}(T)$, if for every $\mathbf{x}, \mathbf{y} \in \Omega$ condition $\|\mathbf{x} - \mathbf{y}\| \leq l$ implies $|DF(\mathbf{x}) - DF(\mathbf{y})| \leq \varepsilon_{DF}$.

Function $G : \Omega \rightarrow \mathbb{R}$ will be called *periodic-like*, $G \in PL_{\Delta}(T)$, if for every $\mathbf{x} \in \Omega$ there exists Δ -periodic function $G_{\mathbf{x}}(\cdot)$ satisfying for every $\mathbf{y} \in B(\mathbf{x}, l) \cap \Omega$ condition $|G(\mathbf{y}) - G_{\mathbf{x}}(\mathbf{y})| \leq \varepsilon_G$. Function $G_{\mathbf{x}}(\cdot)$ will be called a Δ -*periodic approximation* of $G(\cdot)$ near $\Delta(\mathbf{x})$.

A periodic-like function will be called *oscillating*, $G \in PL_{\Delta}^*(T)$, if $G \in PL_{\Delta}(T)$ and if condition $\langle G_{\mathbf{x}} \rangle(\mathbf{x}) = 0$ holds for every $\mathbf{x} \in \Omega_{\Delta}$.

It can be shown that every periodic-like function G can be uniquely decomposed into a sum of slowly varying function G^o and oscillating function G^* , [17]. Under the aforementioned denotations, bearing in mind the meaning of the tolerance parameter and the corresponding tolerance relation, for every $\mathbf{x} \in \Omega_{\Delta}$ we obtain the following approximation formulae

$$(2.1) \quad \begin{aligned} \langle fF \rangle(\mathbf{x}) &\cong \langle f \rangle F(\mathbf{x}), & F \in SV_{\Delta}(T), \\ \langle fG \rangle(\mathbf{x}) &\cong \langle fG_{\mathbf{x}} \rangle(\mathbf{x}), & G \in PL_{\Delta}(T), \end{aligned}$$

where f is an arbitrary integrable Δ -periodic function.

The tolerance averaging of equations with Δ -periodic functional coefficients is based on the assumption that there exists a certain tolerance system T so that formulae (2.1) can be used as approximations in the averaging procedure, [17]. It has to be emphasized that a tolerance system T may be not specified in the course of modelling; all we need is that this system exists. Moreover, the tolerance parameters can be calculated *a posteriori* as certain residuals determining the degree of accuracy of obtained solutions to the special problem under consideration.

3. Modelling approach

Let Ω stand for a region in E^3 occupied by a Δ -periodic elastic-viscoplastic composite solid in its reference configuration. It is assumed that the diameter l of the periodicity cell Δ is sufficiently small when compared to the minimum characteristic length dimension of Ω . Denoting by $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{S}(\mathbf{x}, t)$, the displacement and stress fields, respectively, defined in Ω for every time t , by $\rho(\mathbf{x})$ the mass density field in Ω , and assuming that the body force \mathbf{b} is constant, we obtain the well known form of the equations of motion

$$(3.1) \quad \nabla \cdot \mathbf{S} - \rho \ddot{\mathbf{u}} + \rho \mathbf{b} = 0$$

which have to be satisfied in Ω for every time t together with the known stress continuity conditions on the interfaces between constituents.

To simplify the subsequent considerations we shall restrict ourselves to materials subjected to small strains. The elastic/viscoplastic components are assumed to obey the Huber-Mises yield condition. For the sake of simplicity we shall neglect the hardening effect. Setting

$$\mathbf{S}^D = \mathbf{S} - \frac{1}{3} \mathbf{1} \operatorname{tr} \mathbf{S}, \quad \sigma(\mathbf{S}) = \frac{1}{2} \mathbf{S}^D : \mathbf{S}^D$$

we introduce the concept of the yield surface in the form $\sigma(\mathbf{S}) - k^2 = 0$. Let us denote

$$H(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1 & \text{if } a \geq 0 \end{cases}$$

for every real a . Let μ be a viscous parameter representing the relaxation time and define

$$(3.2) \quad \mathbf{D} = H(\sigma(\mathbf{S}) - k^2) \frac{\sqrt{\sigma(\mathbf{S})} - k}{2\mu\sqrt{\sigma(\mathbf{S})}} \mathbf{S}^D$$

as the viscoplastic strain rate. We shall also assume that viscoelastic component materials are obeying the linear Maxwell's law. Combining together the constitutive equations of elasto/viscoplasticity with the equations of linear viscoelasticity, we obtain

$$(3.3) \quad \varepsilon(\dot{\mathbf{u}}) = \mathbb{A} : \dot{\mathbf{S}} + \mathbb{B} : \mathbf{S} + \mathbf{D}$$

where \mathbb{A} , \mathbb{B} are the compliance tensors describing respectively the elastic and viscous properties of the material. Neglecting in (3.3) the term $\mathbb{B} : \mathbf{S}$ we shall deal with the elastic/viscoplastic component materials. Setting in (3.3) $\mathbf{D} \equiv \mathbf{0}$ we shall describe the behaviour of the linear viscoelastic components. For the solid under consideration \mathbb{A} , \mathbb{B} , μ , k are Δ -periodic functions of \mathbf{x} which attain constant values in every constituent of the composite.

The main aim of this contribution is to derive from Eqs. (3.1)–(3.3), which describe the composite solid on the micro-level, a certain system of equations with constant (averaged) coefficients. The derived equations will be interpreted as describing the composite solid on the macroscopic level. To this end we apply the tolerance averaging approach using the concepts outlined in Sec. 2.

The tolerance averaging is based on the heuristic assumption that in an arbitrary periodicity cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$, which is located away from the boundary of Ω , the displacement $\mathbf{u}(\cdot, t)$ and stress $\mathbf{S}(\cdot, t)$ fields conform to the Δ -periodic structure of the solid under consideration. The above *conformability assumption* states that for every time t fields $\mathbf{u}(\cdot, t)$ and $\mathbf{S}(\cdot, t)$ are periodic-like functions:

$$(3.4) \quad \mathbf{u}(\cdot, t) \in PL_\Delta(T), \quad \mathbf{S}(\cdot, t) \in PL_\Delta(T);$$

the above condition may be violated only near the boundary of the solid. Let us observe that the condition similar to that formulated above is also used in homogenization, cf. [14], p. 204.

Following the results of Sec. 2 we conclude that (3.4) implies the decomposition

$$(3.5) \quad \mathbf{u}(\cdot, t) = \mathbf{w}(\cdot, t) + \mathbf{v}(\cdot, t), \quad \mathbf{w}(\cdot, t) \in SV_{\Delta}(T), \quad \mathbf{v}(\cdot, t) \in PL_{\Delta}^*(T),$$

where $\mathbf{w}(\mathbf{x}, t) = \langle \mathbf{u} \rangle(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, is the averaged displacement field and $\mathbf{v}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, will be referred to as the displacement fluctuation field. Denoting by $\mathbf{v}_{\mathbf{x}}$ and $\mathbf{S}_{\mathbf{x}}$ the Δ -periodic approximations near $\Delta(\mathbf{x})$ of \mathbf{v} and \mathbf{S} , respectively, defining $\mathbf{D}_{\mathbf{x}}$ by (3.2) for $\mathbf{S} = \mathbf{S}_{\mathbf{x}}$ and introducing $\bar{\mathbf{v}}$ and $\bar{\mathbf{S}}$, with $\langle \bar{\mathbf{v}} \rangle = \mathbf{0}$, as arbitrary Δ -periodic test functions, by means of (2.1) we obtain from (3.1) and (3.5) the following variational conditions:

$$(3.6) \quad \begin{aligned} & \langle \nabla \bar{\mathbf{v}} : \mathbf{S}_{\mathbf{x}} \rangle(\mathbf{x}, t) + \langle \varrho \bar{\mathbf{v}} \rangle \cdot \dot{\bar{\mathbf{w}}}(\mathbf{x}, t) + \langle \varrho \bar{\mathbf{v}} \cdot \dot{\bar{\mathbf{v}}}_{\mathbf{x}} \rangle(\mathbf{x}, t) - \langle \varrho \bar{\mathbf{v}} \rangle \cdot \mathbf{b} = 0, \\ & \langle \bar{\mathbf{S}} : \varepsilon(\dot{\mathbf{v}}_{\mathbf{x}}) \rangle(\mathbf{x}, t) + \langle \bar{\mathbf{S}} \rangle : \varepsilon(\dot{\mathbf{w}})(\mathbf{x}, t) \\ & \quad = \langle \bar{\mathbf{S}} : \mathbb{A} : \dot{\mathbf{S}}_{\mathbf{x}} \rangle(\mathbf{x}, t) + \langle \bar{\mathbf{S}} : \mathbb{B} : \mathbf{S}_{\mathbf{x}} \rangle(\mathbf{x}, t) + \langle \bar{\mathbf{S}} : \mathbf{D}_{\mathbf{x}} \rangle(\mathbf{x}, t) \end{aligned}$$

which hold for every $\mathbf{x} \in \Omega_{\Delta}$ provided that \mathbf{x} is not situated near the boundary $\partial\Omega$ of Ω .

In order to apply the second one of the conformability assumptions (3.4), we introduce a partition of Δ into a set of n not intersecting elements (regions) Δ_a , $a = 1, \dots, n$, $\bar{\Delta} = \cup \bar{\Delta}_a$, so that every element Δ_a is homogeneous, i.e., it consists of only one constituent of the composite. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the orthonormal basis in E^3 and let us denote by Λ the Bravais lattice $\Lambda = \{\mathbf{z} \in E^3 : \mathbf{z} = n_1 l_1 \mathbf{e}_1 + n_2 l_2 \mathbf{e}_2 + n_3 l_3 \mathbf{e}_3; n_i = 0, \pm 1, \pm 2, \dots\}$. Define

$$\Xi^a = \{\mathbf{y} \in \Delta_a + \mathbf{z}; \mathbf{z} \in \Lambda\}, \quad \Omega^a = \Xi^a \cap \Omega$$

and let $\eta^a(\cdot)$ be the characteristic function of Ξ^a , $a = 1, \dots, n$. Obviously, $\eta^a(\cdot)$ is a Δ -periodic function. Taking into account that $\mathbf{S}(\cdot, t)$ is a periodic-like function, we shall introduce n sufficiently smooth functions $\mathbf{S}^a(\cdot, t)$ defined in Ω which are slowly varying and every $\mathbf{S}^a(\mathbf{z}, t)$, $\mathbf{z} \in \Lambda \cap \Omega_{\Delta}$, is a mean value of $\mathbf{S}(\cdot, t)$ in $\Delta(\mathbf{z}) \cap \Xi^a$. The subsequent considerations will be based on the extra modelling assumption that in the course of averaging procedure we can use the approximations

$$\mathbf{S}_{\mathbf{x}}(\mathbf{y}, t) \cong \eta^a(\mathbf{y}) \mathbf{S}^a(\mathbf{x}, t); \quad \mathbf{y} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \Omega_{\Delta},$$

where $\mathbf{S}_{\mathbf{x}}(\cdot, t)$ is a Δ -periodic approximation of $\mathbf{S}(\cdot, t)$ near $\Delta(\mathbf{x})$. This approximation holds with a sufficient accuracy provided that the partition of Δ into

elements Δ_a , $a = 1, \dots, n$, is sufficiently fine. Fields $\mathbf{S}^a(\cdot, t)$ describe the stresses on the micro-level but, as slowly varying functions, they will also describe a behaviour of the composite in the framework of a proposed macroscopic model. Following [14], p. 253, we recall that in modelling problems under consideration it is not possible to eliminate entirely the microscopic description from the macroscopic one.

Fields \mathbf{S}^a will be referred to as the mean local stresses. Substituting into (3.6) $\mathbf{S}_x(\mathbf{y}, t) = \eta^a(\mathbf{y})\mathbf{S}^a(\mathbf{x}, t)$, $\mathbf{y} \in \Delta(\mathbf{x})$ and $\bar{\mathbf{S}} = \eta^a(\mathbf{y})\mathbf{C}^a$ where \mathbf{C}^a are arbitrary constant second-order tensors, we obtain

$$\begin{aligned} \langle \eta^a \nabla \bar{\mathbf{v}} \rangle : \mathbf{S}^a(\mathbf{x}, t) + \langle \varrho \bar{\mathbf{v}} \rangle \cdot \ddot{\mathbf{w}}(\mathbf{x}, t) + \langle \varrho \bar{\mathbf{v}} \cdot \dot{\mathbf{v}}_x \rangle(\mathbf{x}, t) - \langle \varrho \bar{\mathbf{v}} \rangle \cdot \mathbf{b} &= 0, \\ (3.7) \quad \langle \eta^a \eta^b \mathbb{A} \rangle : \dot{\mathbf{S}}(\mathbf{x}, t) + \langle \eta^a \eta^b \mathbb{B} \rangle : \mathbf{S}(\mathbf{x}, t) + \langle \eta^a \eta^b \mathbb{D}^b \rangle \mathbf{D}^b(\mathbf{x}, t) \\ &= \langle \eta^a \rangle \varepsilon(\dot{\mathbf{w}})(\mathbf{x}, t) + \langle \eta^a \varepsilon(\dot{\mathbf{v}}_x) \rangle(\mathbf{x}, t) \end{aligned}$$

where (no summation over b !)

$$(3.8) \quad \mathbf{D}^b = H(\sigma(\mathbf{S}^b) - k_b) \frac{\sqrt{\sigma(\mathbf{S}^b)} - k_b}{2\mu_b \sqrt{\sigma(\mathbf{S}^b)}} (\mathbf{S}^b)^D$$

and moduli k_b , μ_b are related to the material components occupying part Ω^b of the region Ω . Fields \mathbf{D}^b will be referred to as the mean local viscoplastic strain rates. Equation (3.7)₁ represents the variational condition which has to hold for every oscillating Δ -periodic test function $\bar{\mathbf{v}}$. By means of the approximation $\mathbf{S} = \eta^a \mathbf{S}^a$ and after restricting the domain Ω of (3.1) to an arbitrary but fixed cell $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$, we obtain the averaged form of equations of motion

$$(3.9) \quad \langle \eta^a \rangle \nabla \cdot \mathbf{S}^a(\mathbf{x}, t) - \langle \varrho \rangle \cdot \ddot{\mathbf{w}}(\mathbf{x}, t) - \langle \varrho \dot{\mathbf{v}}_x \rangle(\mathbf{x}, t) + \langle \varrho \rangle \mathbf{b} = \mathbf{0}.$$

Equations (3.7)–(3.9) constitute the system of equations for averaged displacements \mathbf{w} , mean local stresses \mathbf{S}^a and displacement fluctuations \mathbf{v} (which in every $\Delta(\mathbf{x})$, $\mathbf{x} \in \Omega_\Delta$, are represented by their local periodic approximations \mathbf{v}_x). The above equations have a physical sense only if $\mathbf{w}(\cdot, t)$, $\mathbf{S}^a(\cdot, t)$ are slowly-varying functions.

In order to obtain from (3.7)–(3.9) the macroscopic model of the composite medium under consideration we apply the procedure used in [17]. To this end we shall look for the approximate solution to the periodic variational problem (3.7) for \mathbf{v}_x in the form

$$(3.10) \quad \mathbf{v}_x(\mathbf{y}, t) = h^A(\mathbf{y})\mathbf{v}^A(\mathbf{x}, t), \quad \mathbf{y} \in \Delta(\mathbf{x})$$

where $h^A(\cdot)$ are the postulated a priori linear independent Δ -periodic shape functions and $\mathbf{v}^A(\cdot)$ are new unknowns which will be referred to as fluctuation

variables. Because $\mathbf{v}(\cdot, t) \in PL_{\Delta}^*(T)$, then $\mathbf{v}^A(\cdot, t)$ have to be slowly varying functions. It is assumed that the shape functions lead to the positive definite matrix $\langle \rho h^A h^B \rangle$ and satisfy conditions $\langle h^A \rangle = 0$, $h^A \in O(l)$, $l \nabla h^A \in O(l)$. Let us substitute the right-hand sides of (3.10) into (3.7), (3.9). Using the orthogonalization method we substitute $\bar{\mathbf{v}} = h^A \mathbf{c}^A$ into (3.7)₁; here \mathbf{c}^A are arbitrary independent constant vectors. After simple manipulations, we obtain finally

$$(3.11) \quad \begin{aligned} \langle \rho \rangle \ddot{\mathbf{w}}(\mathbf{x}, t) + \langle \rho h^A \rangle \ddot{\mathbf{v}}^A(\mathbf{x}, t) - \langle \eta^a \rangle \nabla \cdot \mathbf{S}^a(\mathbf{x}, t) - \langle \rho \rangle \mathbf{b} &= \mathbf{0}, \\ \langle \rho h^A h^B \rangle \ddot{\mathbf{v}}^B(\mathbf{x}, t) + \langle \rho h^A \rangle \ddot{\mathbf{w}}(\mathbf{x}, t) + \langle \eta^a \nabla h^A \rangle \cdot \mathbf{S}^a(\mathbf{x}, t) - \langle \rho h^A \rangle \mathbf{b} &= \mathbf{0}, \\ \langle \eta^a \eta^b \mathbb{A} \rangle : \dot{\mathbf{S}}^b(\mathbf{x}, t) + \langle \eta^a \eta^b \mathbb{B} \rangle : \mathbf{S}^b(\mathbf{x}, t) + \langle \eta^a \eta^b \rangle \mathbf{D}^b(\mathbf{x}, t) \\ &\quad - \langle \eta^a \rangle \varepsilon(\dot{\mathbf{w}})(\mathbf{x}, t) - \text{sym}(\langle \eta^a \nabla h^A \rangle \otimes \dot{\mathbf{v}}^A) = \mathbf{0} \end{aligned}$$

where the mean local viscoplastic strain rates \mathbf{D}^b are defined by condition (3.8). It has to be remembered that fields $\mathbf{S}^a(\cdot, t)$, $\mathbf{D}^a(\cdot, t)$ are defined in Ω but have a physical meaning only in Ω^a .

Formulae (3.11) together with (3.8) constitute a system of relations for unknown averaged displacements \mathbf{w} , fluctuation variables \mathbf{v}^A and mean local stresses \mathbf{S}^a . The aforementioned equations have constant coefficients and can be treated as representing a certain *averaged (macroscopic) model of the composite made of elastic/viscoplastic and/or viscoelastic components*. This model has a physical sense if all unknowns for every instant t are slowly-varying functions:

$$(3.12) \quad \mathbf{w}(\cdot, t) \in SV_{\Delta}(T), \quad \mathbf{v}^A(\cdot, t) \in SV_{\Delta}(T), \quad \mathbf{S}^a(\cdot, t) \in SV_{\Delta}(T).$$

The above conditions can be verified only *a posteriori*, i.e., after obtaining a solution to the problem under consideration. In this way we can evaluate on the basis of (3.12) the tolerance parameters related to functions \mathbf{w} , \mathbf{v}^A , \mathbf{S}^a and their derivatives, and hence to determine residuals of approximation for the derived solution. The characteristic feature of the derived model is that it describes the effect of the microstructure size on the macroscopic dynamic behaviour of a solid due to terms $\langle \rho h^A h^B \rangle \in O(l^2)$, $\langle \rho h^A \rangle \in O(l)$ in (3.11). It has to be remembered that the length-scale effect for the problem under consideration is also due to the presence of viscous parameters μ_b in (3.8).

The form and number of equations (3.11) depends on the form and number of shape functions h^A and on the partition of the cell Δ into elements Δ_a . It means that the derived model can be formulated on different levels of accuracy. For a discussion of the problem of finding functions h^A the reader is referred to [17].

4. Special cases

Setting $\mu_a \rightarrow 0$ for $a = 1, \dots, n$ and neglecting in (3.11) the terms involving \mathbb{B} we obtain equations of a composite solid with elastic-perfectly plastic constituents. In this case instead of (3.8) we obtain

$$\mathbf{D}^b = H(\sigma(\mathbf{S}^b) - k_b) \mathbf{D}_p^b$$

where \mathbf{D}_p^b is a rate of the plastic strain in Ω^b . Let us observe that, in contrast to homogenization where we deal with one averaged yield condition for a macroscopic stress, [13,14], here we have separate yield conditions

$$\sigma(\mathbf{S}^a(\mathbf{x}, t)) - k_a = 0$$

for mean local stresses related to different parts Ω^a of Ω . Obviously, if parts Ω^a and Ω^b are occupied by one elastic/viscoplastic constituent then $k_a = k_b$.

Let $\mathbf{S}^a(\mathbf{x}, t) < k_a$, $a = 1, \dots, n$, hold for every $\mathbf{x} \in \Omega$ and every time t . In this case the mean local viscoplastic strain rates \mathbf{D}^a disappear. For the time being let us also neglect in (3.11)₃ the terms involving \mathbb{B} . Under notations (in the definition of ξ^a no summation over $a!$):

$$\xi^a = \eta^a \langle \eta^a \rangle^{-1}, \quad \mathbf{G}^{AB} = \langle \eta^a \nabla h^A \rangle \cdot \langle \xi^a \xi^b \mathbb{C} \rangle \cdot \langle \eta^b \nabla h^B \rangle,$$

where \mathbb{C} is a tensor of elastic moduli, we can eliminate from (3.11) mean local stresses \mathbf{S}^a . After simple manipulations we obtain

$$\langle \rho \rangle \ddot{\mathbf{w}} + \langle \rho h^A \rangle \ddot{\mathbf{v}}^A - \langle \eta^a \rangle \nabla \cdot [\langle \xi^a \mathbb{C} \rangle : \varepsilon(\mathbf{w}) + \langle \xi^a \xi^b \mathbb{C} \rangle : \text{sym}(\langle \eta^b \nabla h^A \rangle \otimes \mathbf{v}^A)] - \langle \rho \rangle \mathbf{b} = \mathbf{0},$$

$$\langle \rho h^A h^B \rangle \ddot{\mathbf{v}}^B + \langle \rho h^A \rangle \ddot{\mathbf{w}} + \mathbf{G}^{AB} \cdot \mathbf{v}^B - \langle \eta^a \nabla h^A \rangle \cdot \langle \xi^a \mathbb{C} \rangle : \varepsilon(\mathbf{w}) - \langle \rho h^A \rangle \mathbf{b} = \mathbf{0}.$$

It can be shown that the above equations, for a sufficiently fine partition of Δ into Δ_a , lead to the equations obtained in [16] for the macroscopic model of Δ -periodic linear elastic composites.

For a homogeneous solid $\langle \rho h^A \rangle = \rho \langle h^A \rangle = 0$, $\langle \eta^1 \nabla h^A \rangle + \langle \eta^2 \nabla h^A \rangle + \dots + \langle \eta^n \nabla h^A \rangle = \langle \nabla h^A \rangle = 0$ and introducing a sufficiently small cell Δ (which for a homogeneous solid may be taken as infinitesimal) we can also assume that $\mathbf{S} = \eta^a \mathbf{S}^a$ and $\mathbf{D} = \eta^a \mathbf{D}^a$ are slowly-varying functions. In this case from equations (3.11)₁, (3.11)₃ we derive equations (3.1) and (3.3), respectively, and under homogeneous initial conditions equations (3.11)₂ yield $\mathbf{v}^B = \mathbf{0}$.

5. Illustrative example

The general model equations (3.11) with conditions (3.8) will be now illustrated and discussed on a simple example of the uniaxial stress in a rod made of two periodically distributed materials, see Fig. 1. In this case equations (3.1), (3.3), after neglecting body forces and denoting $(\cdot)' = \partial(\cdot)/\partial x$, can be reduced to the form

$$(5.1) \quad \begin{aligned} s'(x, t) - \rho(x)\ddot{u}(x, t) &= 0, \\ \dot{u}'(x, t) &= \frac{1}{E(x)}\dot{s}(x, t) + B(x)s(x, t) + d(x, t) \end{aligned}$$

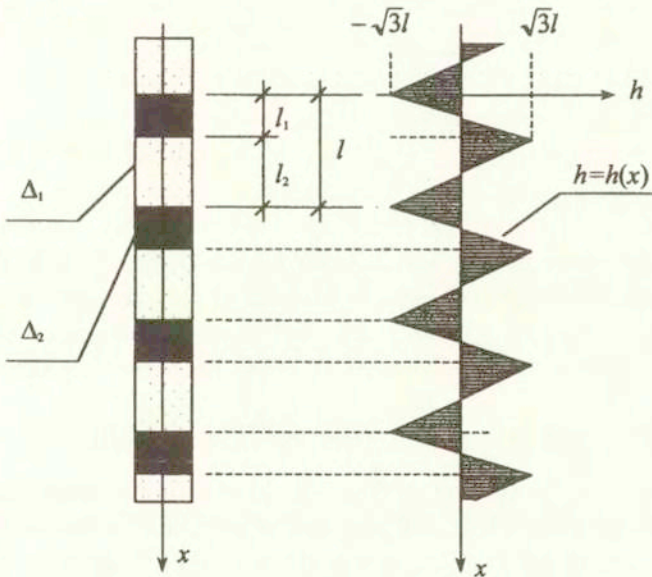


FIG. 1. Scheme of the rod and diagram of the shape function.

and equation (3.2) yields

$$(5.2) \quad d = (H(|s| - \sqrt{3}k(x)) \frac{1}{3\mu(x)} (s - \sqrt{3}k(x)\text{sgn } s)$$

The mass density $\rho(x)$, Young's modulus $E(x)$ and moduli $B(x)$, $k(x)$, $\mu(x)$ are assumed to attain the constant values ρ_1 , E_1 , B_1 , k_1 , μ_1 and ρ_2 , E_2 , B_2 , k_2 , μ_2 , respectively, in the intervals of the x -axis with lengths l_1 , l_2 , see Fig. 1. The cell Δ

now reduces to the interval $(-l/2, l/2)$, $l = l_1 + l_2$. Let us introduce a partition of this cell into two elements $\Delta_1 = (-l/2, -l/2 + l_1)$, $\Delta_2 = (-l/2 + l_1, l/2)$. Following [17] we introduce only one shape function $h = h(x)$, the diagram of which is shown on the right-hand side of Fig. 1. In this case we deal with one fluctuation variable $v = v(x, t)$. Let us also define $\nu_1 = l_1/l$, $\nu_2 = l_2/l$ and denote by s_1, s_2 the mean local stresses. Under the above notation, the model equations (3.11) will take the form

$$\begin{aligned} \langle \rho \rangle \ddot{w}(x, t) - \nu_1 s_1'(x, t) - \nu_2 s_2'(x, t) &= 0, \\ l^2 \langle \rho \rangle \ddot{v}(x, t) + 2\sqrt{3}[s_1(x, t) - s_2(x, t)] &= 0, \\ (5.3) \quad \frac{\nu_1}{E_1} \dot{s}_1(x, t) + \nu_1 B_1 s_1(x, t) + \nu_1 d_1(x, t) - \nu_1 \dot{w}'(x, t) - 2\sqrt{3}\dot{v}(x, t) &= 0, \\ \frac{\nu_2}{E_2} \dot{s}_2(x, t) + \nu_2 B_2 s_2(x, t) + \nu_2 d_2(x, t) - \nu_2 \dot{w}'(x, t) + 2\sqrt{3}\dot{v}(x, t) &= 0 \end{aligned}$$

and condition (3.8) for the uniaxial stress will be given by

$$(5.4) \quad d_\alpha = \frac{1}{3\mu_\alpha} H(|s_\alpha| - \sqrt{3}k_\alpha)(s_\alpha - \sqrt{3}k_\alpha \operatorname{sgn} s_\alpha), \quad \alpha = 1, 2.$$

Equations (5.3), (5.4) constitute the proposed macroscopic model for problems described by equations (5.1), (5.2) provided that $B_1 d_1(\cdot, t) \equiv 0$, $B_2 d_2(\cdot, t) \equiv 0$. Bearing in mind (3.5), the total displacements $u(x, t)$ are now described by the formula $u(x, t) = w(x, t) + h(x)v(x, t)$. We have to remember that by means of (4.1), the solutions to equations (5.3), (5.4) have the physical sense only if conditions

$$w(\cdot, t), v(\cdot, t), s_1(\cdot, t), s_2(\cdot, t) \in SV_\Delta(T)$$

hold for every time t . We have stated in Sec. 4 that the proposed model describes the effect of the microstructure cell size on the overall solid behaviour due to the presence of the length parameter l in equation (5.3)₂. Neglecting the microstructural term $l^2 \langle \rho \rangle \ddot{v}$ we obtain from (5.3) a simplified model in which $s_1 = s_2 = s$. For this simplified model, equation (5.3)₁ takes the form

$$(5.5) \quad \langle \rho \rangle \ddot{w}(x, t) - s'(x, t) = 0.$$

After denotations

$$E^{\text{eff}} = \left(\frac{\nu_1}{E_1} + \frac{\nu_2}{E_2} \right)^{-1}, \quad \langle B \rangle = \nu_1 B_1 + \nu_2 B_2, \quad \alpha = E^{\text{eff}} \langle B \rangle$$

we obtain the following equation for the stress field

$$(5.6) \quad \dot{s}(x, t) + \alpha s(x, t) + E^{\text{eff}}[\nu_1 d_1(x, t) + \nu_2 d_2(x, t)] = E^{\text{eff}} \dot{w}'(x, t)$$

where

$$(5.7) \quad \begin{aligned} d_1 &= \frac{1}{3\mu_1} H(|s| - \sqrt{3}k_1)(s - \sqrt{3}k_1 \operatorname{sgn} s), \\ d_2 &= \frac{1}{3\mu_2} H(|s| - \sqrt{3}k_2)(s - \sqrt{3}k_2 \operatorname{sgn} s); \end{aligned}$$

and constant E^{eff} in (5.6) is the well-known effective value of the Young modulus in the uniaxial stress. In this case we have arrived at the system of equations (5.5), (5.6) for $w(\cdot)$ and $s(\cdot)$, where the partial viscoplastic strain rates are given by conditions (5.7). Neglecting in (5.6) the term involving α we pass to the simplified macroscopic model of the problem under consideration for an elastic/viscoplastic material. This problem is governed by equation (5.5) and

$$\dot{s}(x, t) - E^{\text{eff}} \dot{w}'(x, t) + E^{\text{eff}} [\nu_1 d_1(x, t) + \nu_2 d_2(x, t)] = 0$$

together with conditions (5.7). For viscoelastic composites $d_1 = d_2 \equiv 0$ and equation (5.6) yields

$$(5.8) \quad s(x, t) - s(x, 0) = E^{\text{eff}} [w'(x, t) - w'(x, 0) - \alpha e^{-\alpha t} \int_0^t w'(x, \tau) e^{\alpha \tau} d\tau].$$

Substituting the right-hand side of (5.8) into (5.5) we obtain the equation for the averaged displacement field $w(\cdot)$. The above results hold only under the assumption that the microstructural term $l^2 \langle \rho \rangle \ddot{v}(x, t)$ in (5.3) is neglected.

If the microstructural term $l^2 \langle \rho \rangle \ddot{v}(x, t)$ in (5.3) is not neglected then $s_1 \neq s_2$ and for viscoelastic composites, under notations

$$\alpha_1 = E_1 B_1, \quad \alpha_2 = E_2 B_2,$$

we obtain from (5.3):

$$(5.9) \quad \begin{aligned} s_1(x, t) - s_1(x, 0) &= E_1 \left[w'(x, t) - w'(x, 0) + \frac{2\sqrt{3}}{\nu_1} v(x, t) - \frac{2\sqrt{3}}{\nu_1} v(x, 0) \right. \\ &\quad \left. + \alpha_1 e^{-\alpha_1 t} \int_0^t (w'(x, \tau) + \frac{2\sqrt{3}}{\nu_1} v(x, \tau)) e^{\alpha_1 \tau} d\tau \right], \\ s_2(x, t) - s_2(x, 0) &= E_2 \left[w'(x, t) - w'(x, 0) - \frac{2\sqrt{3}}{\nu_2} v(x, t) + \frac{2\sqrt{3}}{\nu_2} v(x, 0) \right. \\ &\quad \left. + \alpha_2 e^{-\alpha_2 t} \int_0^t (w'(x, \tau) - \frac{2\sqrt{3}}{\nu_2} v(x, \tau)) e^{\alpha_2 \tau} d\tau \right]. \end{aligned}$$

Substituting the right-hand sides of equations (5.9) into equations (5.3)₁ and (5.3)₂ we arrive at the system of two governing equations for $w(\cdot)$ and $v(\cdot)$. For

elastic materials $\alpha_1 = \alpha_2 = 0$ and the aforementioned system of equation can be reduced to the form

$$\begin{aligned} \langle \varrho \rangle \ddot{w}(x, t) - \langle E \rangle w''(x, t) - 2\sqrt{3}(E_2 - E_1)v(x, t) &= 0, \\ l^2 \langle \varrho \rangle \ddot{v}(x, t) + 2\sqrt{3} \left(\frac{E_2}{\nu_2} + \frac{E_1}{\nu_1} \right) v - 2\sqrt{3}(E_2 - E_1)w'(x, t) &= 0 \end{aligned}$$

which coincides with the known result obtained in [16].

6. Conclusions

We close the paper with a summary of new results and information on the macroscopic modelling of micro-periodic elastic/viscoplastic and linear viscoelastic composites.

1. The proposed macroscopic model of the micro-periodic composites made of elastic/viscoplastic and/or linear viscoelastic components was obtained in the form of equations (3.11) together with conditions (3.8). The above equations involve exclusively constant coefficients. The characteristic feature of the proposed model is that it describes the effect of microstructure size on the dynamic overall behaviour of a composite solid (the microstructure length-scale effect).
 2. The aforementioned microstructure length-scale effect takes place only in dynamic problems, i.e., terms describing this effect for the quasi-stationary problems drop out from the governing equations (3.11) of the macroscopic model.
 3. The proposed model introduces the concept of mean local stresses and mean local viscoplastic strain rates into the macroscopic description of a micro-periodic composite and hence makes it possible to formulate constitutive equations on the macroscopic level independently for every material constituent, see equations (3.11)₃, together with condition (3.8).
 4. The proposed modelling approach leads to certain *a posteriori* estimates of solutions which can be derived in every special problem from conditions (3.12).
 5. For the sake of simplicity all considerations have been based on the simplest form (3.2), (3.3) of constitutive equations for elastic/viscoplastic materials. However, the modelling approach outlined in the paper can be easily extended to general constitutive equations proposed by PERZYNA in [6-9].
- The main drawback of the proposed model lies in a possibly large number of unknown fields $\mathbf{v}^A(\cdot)$, $A = 1, \dots, N$, and $\mathbf{S}^a(\cdot)$, $a = 1, \dots, n$, involved in the macroscopic description of the problem in the framework of the model equations (3.11).

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