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Existence and uniqueness of the solution in the frequency domain for the reflection-transmission problem in a viscoelastic layer

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EXISTENCE AND UNIQUENESS for the reflection-transmission process originated in a viscoelastic solid layer are investigated. Wave propagation is framed within the Fourier-transform domain and the oblique incidence is modelled by a factor involving a transverse wave vector. The backward-forward propagation in the axial direction is ascertained through the sign of an energy flux. Next, a connection is established between the energy flux and an Hermitian matrix whose eigenvalues are half positive and half negative. The proof is given that if the matrix has two diagonal blocks, one of which is positive definite and the other is negative definite, the solution to the reflection-transmission problem exists and is unique. The condition on the blocks is found to hold, e.g., for obliquely propagating homogeneous waves in anisotropic elasticity or normally propagating waves in isotropic viscoelasticity.

1. Introduction

THE AIM of this paper is to establish the existence and uniqueness of the solution to the reflection-transmission (RT) problem associated with the oblique incidence on a stratified multilayer sandwiched between two homogeneous half-spaces. The layers and the half-spaces are viscoelastic anisotropic solids. The material properties are constant in the two half-spaces, depend on the depth inside the layers, and may suffer jump discontinuities at the plane interfaces.

Wave propagation in stratified media is investigated in connection with the Helmholtz equation, the Schrödinger equation, the one-dimensional equation of elasticity, the Maxwell equations of electromagnetism. References [1] to [7] provide interesting approaches and results within the direct and the inverse scattering. Nevertheless, it seems that no results are given concerning the existence and uniqueness for the RT problem in dissipative anisotropic solids. Moreover, dissipation and anisotropy require that a system of six first-order ordinary differ-

ential equations should be considered instead of a single second-order or a pair of first-order equations.

Mathematically, there are three remarkable features associated with the RT problem. First, at least at one interface both incident (known) and reflected (unknown) waves occur. This prevents the RT problem from being a genuine boundary-value problem. Secondly, the solutions to the governing equations in the homogeneous half-spaces are required to produce a basis for the representation of the incoming and outgoing waves. Thirdly, a characterization is in order for the direction of propagation of a wave solution in homogeneous regions. These features are strictly interrelated.

Existence and uniqueness for the RT problem is investigated in [8] for an elastic layer and normal incidence in the time domain and in [9] for oblique incidence on the interface between two homogeneous viscoelastic half-spaces in the frequency domain. The RT problem is investigated in electromagnetism [10, 11] but the scheme does not seem to apply in mechanics. The results obtained for the one-dimensional Schrödinger equation on the line [3] cannot be generalized directly, e.g. to dissipative and anisotropic materials.

Here we take advantage of some suggestions arising from two recent papers of ours. In [12] uniqueness is proved to follow from an appropriate form of the boundary conditions for the layer, which express the outgoing character of reflected and transmitted waves. This aspect however has to be reconsidered because [12] deals with elastic solids. In [9] viscoelastic solids are considered and the direction of a wave is associated with the sign of an energy flux.

In this paper, the wave propagation in viscoelastic solids is framed within the Fourier-transform domain, and the backward-forward propagation in the axial direction is ascertained through the sign of an appropriate energy flux $\mathcal{F}$. The conceptual improvement in this approach is the monotone character of $\mathcal{F}$ which follows from the balance of energy. The oblique incidence is modelled by a factor involving the transverse wave vector $\mathbf{k}_\perp$. Next a connection is established between $\mathcal{F}$ and a Hermitian matrix $\mathbf{\Phi}$ whose eigenvalues are half positive and half negative. We prove that, whenever the matrix $\mathbf{\Phi}$ has two diagonal blocks, one of which is positive definite and one is negative definite, the solution to the RT problem exists and is unique. This means that existence and uniqueness hold, provided in each half-space the fundamental solutions are partitioned in three incoming and three outgoing waves and any linear combination of the triplet preserves the incoming or the outgoing character. As examples, we show that the required property on the blocks of $\mathbf{\Phi}$ holds for obliquely-propagating homogeneous waves in elastic anisotropic solids, or normally-propagating waves in viscoelastic isotropic solids, but does not hold for inhomogeneous (evanescent) waves in elastic solids.

2. Basic framework

Consider a body occupying the whole space domain $\mathbb{R}^3$. Each position in the body is associated with the Cartesian coordinates x, y, z or x_1, x_2, x_3. Denote by $\mathbf{x} = (x, y, z)$ the position vector. Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement at point $\mathbf{x}$ at time t. Also, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the unit vectors of x_1, x_2, x_3 and let ∂_t denote the partial differentiation with respect to t. Body forces are disregarded and the linear approximation is considered. Hence the equation of motion is written in the form

$$\rho \partial_t^2 \mathbf{u} = \nabla \cdot \mathbf{T}, \tag{2.1}$$

where ρ is the mass density and $\mathbf{T}$ is the Cauchy stress tensor.

The body is taken to be anisotropic and linearly viscoelastic. We then write $\mathbf{T}$ in terms of the gradient of displacement, $\nabla \mathbf{u}$, in the form

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{G}_0(\mathbf{x})\, \nabla \mathbf{u}(\mathbf{x}, t) + \int_0^\infty \mathbf{G}'(\mathbf{x}, \eta) \nabla \mathbf{u}(\mathbf{x}, t - \eta) d\eta, \tag{2.2}$$

where the values of $\mathbf{G}_0$ and $\mathbf{G}'$ are fourth-order tensors and $\mathbf{G}'(\mathbf{x}, \eta) = 0$ as $\eta < 0$. In indicial form,

$$T_{ij}(\mathbf{x}, t) = G_{0ijhk} \partial u_k / \partial x_h + \int_0^\infty G'_{ijhk}(\mathbf{x}, \eta) \partial u_k(\mathbf{x}, t - \eta) / \partial x_h \, d\eta.$$

Both $\mathbf{G}_0$ and $\mathbf{G}'$ are required to satisfy the minor and major symmetries, namely

$$G'_{ijhk} = G'_{jihk} = G'_{ijkh} = G'_{hkij}.$$

For any pair of tensors $\mathbf{A}, \mathbf{B}$ we let $\mathbf{A} \cdot \mathbf{B}$ represent the inner product which in components has the form

$$\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}.$$

It is a consequence of thermodynamics [13] that the half-range sine transform of $\mathbf{G}'$,

$$\mathbf{G}'_s(\mathbf{x}, \omega) = \int_0^\infty \mathbf{G}'(\mathbf{x}, \eta) \sin \omega\eta \, d\eta,$$

is definite in the space of symmetric tensors Sym, i.e.

$$\omega \mathbf{E} \cdot \mathbf{G}'_s(\mathbf{x}, \omega) \mathbf{E} < 0, \qquad \forall\, \mathbf{E} \in \mathrm{Sym}, \quad \forall \omega \in \mathbb{R} \setminus \{0\}. \tag{2.3}$$

We let $\mathbf{u}$ depend on t through the factor $\exp(i\omega t)$ or rather address attention to the Fourier transform

$$\mathbf{u}_F(\mathbf{x},\omega) = \int_{-\infty}^{\infty} \mathbf{u}(\mathbf{x},t)\exp(-i\omega t)dt$$

so that

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{u}_F(\mathbf{x},\omega)\exp(i\omega t)d\omega.$$

In this regard we let $\mathbf{u}(\mathbf{x},\cdot), \nabla\mathbf{u}(\mathbf{x},\cdot), \partial_t\mathbf{u}(\mathbf{x},\cdot), \partial_t^2\mathbf{u}(\mathbf{x},\cdot) \in L^1(\mathbb{R})$.

By applying the Fourier transform to (2.2) we find that

$$\mathbf{T}_F(\mathbf{x},\omega) = \mathbf{G}(\mathbf{x},\omega)\nabla\mathbf{u}_F(\mathbf{x},\omega), \tag{2.4}$$

where

$$\mathbf{G}(\mathbf{x},\omega) = \mathbf{G}_0(\mathbf{x}) + \int_0^{\infty} \mathbf{G}'(\mathbf{x},\eta)\exp(-i\omega\eta)\, d\eta.$$

Of course $\mathbf{G}$ inherits the major and minor symmetries from $\mathbf{G}_0$ and $\mathbf{G}'$. Let the subscripts R, I denote the real and imaginary parts, e.g. $\mathbf{G}_R = \Re\mathbf{G}$, $\mathbf{G}_I = \Im\mathbf{G}$. Since $\mathbf{G}_I = -\mathbf{G}'_s$, the restriction of thermodynamic character on $\mathbf{G}'$ can be written as

$$\omega\mathbf{E}\cdot\mathbf{G}_I(\mathbf{x},\omega)\mathbf{E} > 0, \qquad \forall\ \mathbf{E}\in \mathrm{Sym}, \quad \forall\omega\in\mathbb{R}\setminus\{0\}. \tag{2.5}$$

3. Energy flux vector in the frequency domain

We now investigate consequences of (2.1)–(2.5) in the frequency domain. Apply the Fourier transform to (2.1) to obtain

$$i\omega\rho\mathbf{v}_F = \nabla\cdot\mathbf{T}_F, \tag{3.1}$$

where $\mathbf{v} = \partial_t\mathbf{u}$ is the velocity. The dependence of $\mathbf{u}_F, \mathbf{v}_F, \mathbf{T}_F$ on $\mathbf{x}$ and ω is often understood and not written. Let an asterisk denote complex conjugation. Let us multiply (3.1) by $\mathbf{v}_F^*$ to obtain

$$i\omega\rho\mathbf{v}_F\cdot\mathbf{v}_F^* = \nabla\cdot(\mathbf{T}_F\mathbf{v}_F^*) - \mathbf{T}_F\cdot\nabla\mathbf{v}_F^*.$$

The left-hand side is imaginary and hence

$$\Re[\nabla\cdot(\mathbf{T}_F\mathbf{v}_F^*)] = \Re[\mathbf{T}_F\cdot\nabla\mathbf{v}_F^*]. \tag{3.2}$$

We now evaluate $\Re[\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*]$. Since $\nabla \mathbf{v}_F^* = -i\omega \nabla \mathbf{u}_F^*$, upon substitution and use of the symmetries of $\mathbf{G}$ we have

$$\Re[\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*] = \mathbf{E}_R \cdot \omega \mathbf{G}_I(\omega) \mathbf{E}_R + \mathbf{E}_I \cdot \omega \mathbf{G}_I(\omega) \mathbf{E}_I, \tag{3.3}$$

where $\mathbf{E}$ now stands for $\mathrm{Sym}\nabla \mathbf{u}_F$. In viscoelasticity, the positive definiteness of $\omega \mathbf{G}_I(\omega)$, $\omega \neq 0$, yields

$$\Re[\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*] > 0 \quad \forall\, \omega \in \mathbb{R} \setminus \{0\} \tag{3.4}$$

and equality holds at $\omega = 0$. In elasticity $\mathbf{G}_I = 0$ and hence

$$\Re(\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*) = 0, \quad \forall\, \omega \in \mathbb{R}.$$

In both cases we can then write

$$\Re(\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*) \geq 0 \quad \forall\, \omega \in \mathbb{R}; \tag{3.5}$$

equality holds if $\omega = 0$ or in elasticity.

In terms of

$$\mathbf{j} := -\frac{1}{2} \Re[\mathbf{T}_F \mathbf{v}_F^*], \tag{3.6}$$

by (3.2), (3.3) and (3.5) we have

$$\nabla \cdot \mathbf{j} = -\frac{1}{2} [\mathbf{E}_R \cdot \omega \mathbf{G}_I(\omega) \mathbf{E}_R + \mathbf{E}_I \cdot \omega \mathbf{G}_I(\omega) \mathbf{E}_I] \leq 0.$$

Hence, for any region $V \subset \mathbb{R}^3$, with boundary ∂V, the divergence theorem gives

$$0 \geq \int_{\partial V} \mathbf{j} \cdot \mathbf{n}\, da, \tag{3.7}$$

where $\mathbf{n}$ is the unit outward normal to ∂V. If $\omega = 0$, or the solid is elastic, then $\nabla \cdot \mathbf{j} = 0$.

By (3.7) we regard $\mathbf{j}$ as the (real) energy flux vector. This view is motivated also as follows. Consider the strain tensor

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_1(\mathbf{x}) \cos \omega t + \mathbf{E}_2(\mathbf{x}) \sin \omega t.$$

Substitution in (2.2) gives the corresponding stress tensor. Hence, the energy dissipation in the period $[0, \tau]$, $\tau := 2\pi/|\omega|$, yields

$$\int_0^\tau \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) dt = \frac{1}{2} \tau (\mathbf{E}_1 \cdot \mathbf{G}_I \mathbf{E}_1 + \mathbf{E}_2 \cdot \mathbf{G}_I \mathbf{E}_2)$$

thus showing that $(\mathbf{E}_1 \cdot \mathbf{G}_I\mathbf{E}_1 + \mathbf{E}_2 \cdot \mathbf{G}_I\mathbf{E}_2)/2$ is the energy dissipated per unit time. The identification $\mathbf{E}_1 = \mathbf{E}_R$, and $\mathbf{E}_2 = \mathbf{E}_I$ shows that, by (3.7), $\mathbf{j}$ is rightly viewed as the energy flux vector. As a consequence,

$$\mathbf{j} \cdot \mathbf{n} = -\frac{1}{2}\Re[(\mathbf{T}_F\mathbf{n}) \cdot \mathbf{v}_F^*] \tag{3.8}$$

is the energy flux, per unit area and unit time, through a surface with normal $\mathbf{n}$.

The results for the energy flux vector and the energy flux are consistent with the argument given, e.g., in [14], Sec. 2.5, and [15], Sec. 3.4, and [16] where the power, in time-harmonic motions, is taken as the time average, over a time period, of the inner product of the real part of the force and the real part of the velocity.

4. First-order system of equations

Henceforth we let $\mathbf{G}$ and ρ depend on $\mathbf{x}$ through z. The governing equations can then be written as a first-order system, which in turn is essential for the investigation of the energy flux and the proof of uniqueness.

Let $\mathbf{w}(z, \omega)$ take values in $\mathbb{C}^6$ and set

$$\begin{bmatrix} \mathbf{u}_F \\ \mathbf{t}_F \end{bmatrix} = \mathbf{w} \exp(i\mathbf{k}_\perp \cdot \mathbf{x}), \tag{4.1}$$

where $\mathbf{t}_F = \mathbf{T}_F\mathbf{e}_3$ is the traction on the pertinent z=constant plane and $\mathbf{k}_\perp \in \mathbb{C}^3$, $\mathbf{k}_\perp \cdot \mathbf{e}_3 = 0$. The factor $\exp(i\mathbf{k}_\perp \cdot \mathbf{x})$ allows for oblique incidence and traces back to Snell's law.

The equation of motion (2.1) then takes the form of a first-order system of equations

$$\mathbf{w}' = \mathbf{A}\mathbf{w}, \tag{4.2}$$

where $\mathbf{A}(z, \omega) \in \mathbb{C}^{6\times 6}$ and a prime stands for differentiation with respect to z. The matrix $\mathbf{A}$ is determined as follows. For any pair $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$, let $\mathbf{aGb}$ stand for the matrix with components $(\mathbf{aGb})_{hk} = a_p G_{hpqk} b_q$. Hence

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^I & \mathbf{A}^{II} \\ \mathbf{A}^{III} & \mathbf{A}^{IV} \end{bmatrix}$$

where

$$\mathbf{A}^I = -i(\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{G}\mathbf{k}_\perp), \qquad \mathbf{A}^{II} = (\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1},$$

$$\mathbf{A}^{III} = -\rho\omega^2\mathbf{1} + \mathbf{k}_\perp\mathbf{G}\mathbf{k}_\perp - (\mathbf{k}_\perp\mathbf{G}\mathbf{e}_3)(\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{G}\mathbf{k}_\perp), \quad \mathbf{A}^{IV} = (\mathbf{A}^I)^T.$$

Since $\mathbf{A}^{II}$ and $\mathbf{A}^{III}$ are symmetric it follows that, letting

$$\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

we have

$$\mathbf{KA} = (\mathbf{KA})^T. \tag{4.3}$$

The form (4.2) for the governing equations traces back to STROH [17] for elastic materials; cf. [9].

4.1. Representation of w in homogeneous regions

Look at homogeneous regions (half-spaces) where $\mathbf{A}$ is independent of z. We assume that $\mathbf{A}$ is simple and hence that $\mathbf{A}$ has six linearly independent eigenvectors, say $\mathbf{p}_\alpha$, $\alpha = 1, ..., 6$. Denote by $i\sigma_\alpha$ the corresponding eigenvalues. We assume that each σ_α is nonzero. The values of σ_α are determined by solving the eigenvalue problem

$$\mathbf{Ap} = i\sigma\mathbf{p}, \tag{4.4}$$

where $\mathbf{A}$ is parameterized by $\mathbf{k}_\perp$ and ω. In this regard, represent $\mathbf{p}$ as the ordered pair of triplets $[\mathbf{a}, \mathbf{l}]^T$. By using the block form of $\mathbf{A}$, we find that (4.4) is equivalent to

$$[\sigma^2\mathbf{e}_3\mathbf{G}\mathbf{e}_3 + \sigma(\mathbf{e}_3\mathbf{G}\mathbf{k}_\perp + \mathbf{k}_\perp\mathbf{G}\mathbf{e}_3) - \rho\omega^2\mathbf{1} + \mathbf{k}_\perp\mathbf{G}\mathbf{k}_\perp]\mathbf{a} = 0. \tag{4.5}$$

The values of σ are then given by the secular equation

$$\det[-\rho\omega^2\mathbf{1} + \mathbf{k}_\perp\mathbf{G}\mathbf{k}_\perp + \sigma^{\mathbf{2}}\mathbf{e_3}\mathbf{G}\mathbf{e_3} + \sigma(\mathbf{e_3}\mathbf{G}\mathbf{k}_\perp + \mathbf{k}_\perp\mathbf{G}\mathbf{e_3})] = 0. \tag{4.6}$$

Correpondingly, the value of $\mathbf{a}$ is determined by (4.5) and that of $\mathbf{l}$ by

$$\mathbf{l} = i(\mathbf{e}_3\mathbf{G}\mathbf{k}_\perp + \sigma\mathbf{e}_3\mathbf{G}\mathbf{e}_3)\mathbf{a}. \tag{4.7}$$

The integral of (4.2) is written in the form

$$\mathbf{w}(z) = \sum_\alpha c_\alpha\mathbf{p}_\alpha \exp[i\sigma_\alpha(z - z_0)], \tag{4.8}$$

where $c_\alpha, \mathbf{p}_\alpha$ and σ_α are complex-valued and parameterized by ω while z_0 is a reference value of z. The representation (4.8) of $\mathbf{w}$ shows that $\{\mathbf{p}_\alpha \exp[i\sigma_\alpha(z - z_0)]\}$ is a basis for the solution to (4.2). It is worth remarking that the form

$$[\mathbf{u}_F, \mathbf{t}_F]^T(z, \omega) \exp(i\omega t) = \sum_\alpha c_\alpha\mathbf{p}_\alpha \exp[i(\mathbf{k}_\perp \cdot \mathbf{x} + \sigma_\alpha(z - z_0) + \omega t)] \tag{4.9}$$

of the pair $[\mathbf{u}_F, \mathbf{t}_F]^T$ allows the solution to be viewed as a superposition of inhomogeneous waves (cf. [18, 15]). Hence $\mathbf{a}$ can be viewed as the polarization of the displacement and $\mathbf{l}$ as the polarization of the traction.

It is convenient to consider the matrix

$$\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_6],$$

namely the matrix whose columns are the eigenvectors of $\mathbf{A}$. The linear independence of the eigenvectors makes $\mathbf{P}$ invertible. Moreover we have

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda},$$

where

$$\mathbf{\Lambda} = \mathrm{diag}[i\sigma_1, ..., i\sigma_6].$$

Letting

$$\mathbf{c} = [c_1, ..., c_6]^T$$

and

$$\mathbf{E}(z, z_0) = \mathrm{diag}[\exp(i\sigma_1(z - z_0)), ..., \exp(i(z - z_0))],$$

we can write (4.8) in the form

$$\mathbf{w}(z) = \mathbf{P}\mathbf{E}(z, z_0)\mathbf{c}. \tag{4.10}$$

We next make use of the freedom in choosing z_0 so as to simplify the description of the pertinent problem. For distinct eigenvalues, by (4.3) the eigenvectors satisfy the $\mathbf{K}$-orthogonality condition

$$(\mathbf{P}^T\mathbf{K}\mathbf{P})_{\beta\alpha} = \mathbf{p}_\beta^T\mathbf{K}\mathbf{p}_\alpha = \gamma_\alpha\delta_{\alpha\beta}. \tag{4.11}$$

If the eigenvalues are not all distinct, we can use the arbitrariness in the choice of $\{\mathbf{p}_\alpha\}$ by letting (4.11) still hold.

5. Decay of the energy flux

We now show that the energy flux decays while the wave propagates. This property is essential to prove the uniqueness of the solution.

THEOREM 1. *Let $\mathbf{k}_\perp \in \mathbb{R}^3$. The functions $[\mathbf{u}_F, \mathbf{t}_F]^T$ in (4.1), subject to (2.4) and (4.2), and $\mathbf{v}_F = i\omega\mathbf{u}_F$ satisfy the identity*

$$\frac{d}{dz}\Re(\mathbf{t}_F \cdot \mathbf{v}_F^*) = \Re(\mathbf{T}_F \cdot \nabla\mathbf{v}_F^*). \tag{5.1}$$

P r o o f. Let $\mathbf{w} = [\hat{\mathbf{u}}, \hat{\mathbf{t}}]^T$ so that

$$\mathbf{u}_F = \hat{\mathbf{u}} \exp(i\mathbf{k}_\perp \cdot \mathbf{x}), \qquad \mathbf{t}_F = \hat{\mathbf{t}} \exp(i\mathbf{k}_\perp \cdot \mathbf{x}).$$

Since

$$\nabla \mathbf{u}_F = i\mathbf{k}_\perp \otimes \mathbf{u}_F + \mathbf{e}_3 \otimes \mathbf{u}'_F$$

by (2.4) we have

$$\mathbf{t}_F = [i\mathbf{G}(\mathbf{k}_\perp \otimes \mathbf{u}_F) + \mathbf{G}(\mathbf{e}_3 \otimes \mathbf{u}'_F)]\mathbf{e}_3.$$

Also, $\mathbf{v}^*_F = -i\omega \mathbf{u}^*_F$ and

$$\nabla \mathbf{v}^*_F = -\omega \mathbf{k}_\perp \otimes \mathbf{u}^*_F - i\omega \mathbf{e}_3 \otimes \mathbf{u}'^*_F.$$

Consequently by using (2.4), the symmetry of $\mathbf{G}$, and the reality of $\mathbf{k}_\perp$ we have

$$\mathbf{T}_F \cdot \nabla \mathbf{v}^*_F = -\omega(i\mathbf{k}_\perp \otimes \hat{\mathbf{u}} + \mathbf{e}_3 \otimes \hat{\mathbf{u}}') \cdot \mathbf{G}(\mathbf{k}_\perp \otimes \hat{\mathbf{u}}^*) - i\omega(i\mathbf{k}_\perp \otimes \hat{\mathbf{u}} + \mathbf{e}_3 \otimes \hat{\mathbf{u}}') \cdot \mathbf{G}(\mathbf{e}_3 \otimes \hat{\mathbf{u}}'^*).$$

By (4.2) we have

$$\hat{\mathbf{u}}' = -i(\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{G}\mathbf{k}_\perp)\hat{\mathbf{u}} + (\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1}\hat{\mathbf{t}}.$$

Hence, by using again (4.2) we have

$$\hat{\mathbf{t}}' = -\rho\omega^2 \hat{\mathbf{u}} + (\mathbf{k}_\perp \mathbf{G} \mathbf{k}_\perp)\hat{\mathbf{u}} - i(\mathbf{k}_\perp \mathbf{G} \mathbf{e}_3)\hat{\mathbf{u}}'.$$

Consequently,

$$\begin{aligned}\frac{d}{dz}(\mathbf{t}_F \cdot \mathbf{u}^*_F) &= (i\mathbf{k}_\perp \otimes \hat{\mathbf{u}} + \mathbf{e}_3 \otimes \hat{\mathbf{u}}') \cdot \mathbf{G}(\mathbf{e}_3 \otimes \hat{\mathbf{u}}'^*) \\ &\quad - \rho\omega^2 \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^* + (\hat{\mathbf{u}}^* \otimes \mathbf{k}_\perp) \cdot \mathbf{G}(\mathbf{k}_\perp \otimes \hat{\mathbf{u}}) - i(\hat{\mathbf{u}}^* \otimes \mathbf{k}_\perp) \cdot \mathbf{G}(\mathbf{e}_3 \otimes \hat{\mathbf{u}}').\end{aligned}$$

A comparison with the expression of $\mathbf{T}_F \cdot \nabla \mathbf{v}^*_F$ yields

$$i\omega \frac{d}{dz}(\mathbf{t}_F \cdot \mathbf{u}^*_F) + \mathbf{T}_F \cdot \nabla \mathbf{v}^*_F = -i\omega^3 \rho \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^*.$$

The right-hand side is purely imaginary. Hence, taking the real part we have

$$\Re[i\omega \frac{d}{dz}(\mathbf{t}_F \cdot \mathbf{u}^*_F)] = -\Re[\mathbf{T}_F \cdot \nabla \mathbf{v}^*_F]$$

whence the conclusion follows. □

In view of (3.8) we let

$$\mathcal{F} := \mathbf{j} \cdot \mathbf{e}_3 = -\frac{1}{2}\Re[\mathbf{t}_F \cdot \mathbf{v}_F^*] \tag{5.2}$$

be the energy flux in the direction $\mathbf{e}_3$. If $\mathcal{F} > 0$ ($\mathcal{F} < 0$) then energy is flowing, across a z = constant surface, in the positive (negative) z-direction. Let $\hat{\mathcal{F}}$ be the functional such that

$$\mathcal{F}(z) = \hat{\mathcal{F}}(\mathbf{w}(z)).$$

If $\mathcal{F} > 0$ then the pertinent wave $\mathbf{w}$ is said to propagate in the positive z-direction.

COROLLARY 1. The function $\mathcal{F}$ is non-increasing.

P r o o f. By (5.2) and (5.1) we have

$$\mathcal{F}' = -\frac{1}{2}\Re(\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*).$$

The positive definiteness of $\Re(\mathbf{T}_F \cdot \nabla \mathbf{v}_F^*)$ implies that

$$\mathcal{F}' \le 0 \tag{5.3}$$

and equality holds when $\omega = 0$ or in elastic solids ($\mathbf{G}_I = 0$). □

COROLLARY 2. The function $\mathcal{F}$ cannot change the sign.

P r o o f. The result (5.3) means that the energy flux decays while the wave propagates. In this regard consider $\mathcal{F}^2$ and observe that

$$(\mathcal{F}^2)' = 2\mathcal{F}\mathcal{F}'.$$

If $\mathcal{F} > 0$ then the wave propagates forward (z-direction) and $(\mathcal{F}^2)' \le 0$. If, instead, $\mathcal{F} < 0$ then the wave propagates backward ($-z$-direction) and $d\mathcal{F}^2/d(-z) = -(\mathcal{F}^2)' \le 0$. In both cases $\mathcal{F}^2$ decreases as the wave propagates. Since

$$\frac{d\mathcal{F}^2}{dz} \le 0 \quad \text{if} \quad \mathcal{F} > 0 \qquad \text{and} \qquad \frac{d\mathcal{F}^2}{d(-z)} \le 0 \quad \text{if} \quad \mathcal{F} < 0$$

then $\mathcal{F}^2$ decreases in any direction. So, if there is $\bar{z}$ such that $\mathcal{F}(\bar{z}) = 0$ then either $\mathcal{F}(z) \ge 0$ as $z < \bar{z}$ and $\mathcal{F}(z) = 0$ as $z > \bar{z}$, or $\mathcal{F}(z) \le 0$ as $z > \bar{z}$ and $\mathcal{F}(z) = 0$ as $z < \bar{z}$. Hence $\mathcal{F}$ cannot change the sign. □

REMARK 1. If $\mathbf{k}_\perp \in \mathbb{C}^3$ then the result (5.1) is no longer true. Indeed,

$$\begin{aligned} i\omega\frac{d}{dz}(\mathbf{t}_F \cdot \mathbf{u}_F^*) = \exp[-2\Im\mathbf{k}_\perp \cdot \mathbf{x}]\{&-i\omega^3\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^* + i\omega(\mathbf{e}_3 \otimes \hat{\mathbf{u}}'^* \cdot \mathbf{G}(\mathbf{e}_3 \otimes \hat{\mathbf{u}}')) \\ &+ i\omega\hat{\mathbf{u}}^*(\mathbf{k}_\perp\mathbf{G}\mathbf{k}_\perp)\hat{\mathbf{u}} + \omega[\hat{\mathbf{u}}^* \cdot (\mathbf{k}_\perp\mathbf{G}\mathbf{e}_3)\hat{\mathbf{u}}' - \hat{\mathbf{u}} \cdot (\mathbf{k}_\perp\mathbf{G}\mathbf{e}_3)\hat{\mathbf{u}}'^*]\} \end{aligned}$$

and the real part is not (negative) definite. Consequently, the restriction $\mathbf{k}_\perp \in \mathbb{R}^3$ for Theorem 1 and, moreover, for the inequality (5.3) is really necessary. The reality of $\mathbf{k}_\perp$ seems to be a requirement placed by the joint occurrence of the oblique incidence and the unboundedness of the plane interface. This in turn shows that (5.3) is not merely the law of energy conservation. Energy conservation applies to a volume, (5.3) involves a single direction.

6. The energy flux as a Hermitian quadratic form

In order to establish the existence and uniqueness for the RT problem we now investigate some algebraic properties of $\mathcal{F}$ in homogeneous solids.

For later applications where (5.3) is crucial, henceforth we restrict attention to a real-valued $\mathbf{k}_\perp$. The results of this section, though, hold with trivial changes if $\mathbf{k}_\perp$ is complex-valued.

Since $\mathbf{v}_F = i\omega\mathbf{u}_F$, by (5.2) we have

$$\mathcal{F} = -\frac{\omega}{2}\Im(\mathbf{t}_F \cdot \mathbf{u}_F^*) = \frac{i\omega}{4}(\mathbf{t}_F \cdot \mathbf{u}_F^* - \mathbf{t}_F^* \cdot \mathbf{u}_F).$$

Hence, letting

$$\boldsymbol{\mathcal{I}} = \frac{i\omega}{4}\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$$

we can write $\mathcal{F}$ in the form

$$\mathcal{F} = [\mathbf{u}_F^*, \mathbf{t}_F^*]\,\boldsymbol{\mathcal{I}}\begin{bmatrix} \mathbf{u}_F \\ \mathbf{t}_F \end{bmatrix}$$

and also

$$\mathcal{F}(z) = \mathbf{w}^\dagger(z)\,\boldsymbol{\mathcal{I}}\,\mathbf{w}(z).$$

Letting

$$\boldsymbol{\Phi} = \mathbf{P}^\dagger\boldsymbol{\mathcal{I}}\,\mathbf{P}, \tag{6.1}$$

by (4.10) we obtain

$$\mathcal{F}(z) = \mathbf{c}^\dagger\mathbf{E}^\dagger(z, z_0)\boldsymbol{\Phi}\mathbf{E}(z, z_0)\mathbf{c}.$$

It is convenient to represent the α-th eigenvector $\mathbf{p}_\alpha$ as the ordered pair of triplets $[\mathbf{a}_\alpha, \mathbf{l}_\alpha]$. Accordingly we have

$$\Phi_{\alpha\beta} = \frac{i\omega}{4}(\mathbf{a}_\alpha^* \cdot \mathbf{l}_\beta - \mathbf{l}_\alpha^* \cdot \mathbf{a}_\beta). \tag{6.2}$$

The energy flux $\mathcal{F}$ can then be written as

$$\mathcal{F}(z) = \sum_{\alpha,\beta} c_\alpha^* c_\beta \Phi_{\alpha\beta} \exp[i(\sigma_\beta - \sigma_\alpha^*)(z - z_0)].$$

In particular,

$$\mathcal{F}(z_0) = \sum_{\alpha,\beta} c_\alpha^* c_\beta \Phi_{\alpha\beta}. \tag{6.3}$$

Since $\boldsymbol{\mathcal{I}}$ is Hermitian, $\boldsymbol{\mathcal{I}} = \boldsymbol{\mathcal{I}}^\dagger$, then by (6.1) $\boldsymbol{\Phi}$ proves to be Hermitian too,

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^\dagger.$$

The eigenvalues $\lambda_1, ..., \lambda_6$ of $\boldsymbol{\mathcal{I}}$ are

$$\lambda_{1,2,3} = \omega/4, \qquad \lambda_{4,5,6} = -\omega/4.$$

This follows at once by

$$\det[\boldsymbol{\mathcal{I}} - \lambda \mathbf{1}] = (\lambda^2 - \omega^2/16)^3$$

or because

$$\boldsymbol{\mathcal{I}}\,\boldsymbol{\mathcal{I}} = \frac{\omega^2}{16}\mathbf{1}.$$

Since $\boldsymbol{\Phi}$ is Hermitian, the eigenvalues $\phi_1, ..., \phi_6$ say, are real. Also, $\boldsymbol{\Phi}$ is congruent with $\boldsymbol{\mathcal{I}}$. Now, two Hermitian matrices (as $\boldsymbol{\Phi}$ and $\boldsymbol{\mathcal{I}}$) are congruent if and only if they have the same rank and the same number (counting multiplicities) of positive and negative eigenvalues (see [19], p. 185). Hence $\boldsymbol{\Phi}$ has three positive and three negative eigenvalues, say

$$\phi_{1,2,3} > 0, \qquad \phi_{4,5,6} < 0. \tag{6.4}$$

For convenience we represent $\boldsymbol{\Phi}$ in terms of 3×3 blocks,

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}^f & \tilde{\boldsymbol{\Phi}} \\ \tilde{\boldsymbol{\Phi}}^\dagger & \boldsymbol{\Phi}^b \end{bmatrix}. \tag{6.5}$$

We assume that the diagonal blocks are definite, $\boldsymbol{\Phi}^f$ is positive definite and $\boldsymbol{\Phi}^b$ is negative definite. Section 9 shows how such an assumption is significant. The property holds for oblique incidence of homogeneous waves in elastic half-spaces and for normal incidence in isotropic viscoelastic half-spaces. It does not hold if inhomogeneous waves $\sigma \in \mathbb{C}$ occur.

According to the assumption

$$\mathbf{w}(z) = \sum_{h=1}^{3} c_h \mathbf{p}_h \exp(i\sigma_h(z - z_0)) \quad \text{or} \quad \mathbf{w}(z) = \sum_{h=4}^{6} c_h \mathbf{p}_h \exp(i\sigma_h(z - z_0)),$$

by (6.3) we have

$$\mathcal{F}(z_0) = \sum_{h,k=1}^{3} c_h^* c_k \Phi_{hk} > 0 \quad \text{or} \quad \mathcal{F}(z_0) = \sum_{h,k=4}^{6} c_h^* c_k \Phi_{hk} < 0 \tag{6.6}$$

for every non-trivial triplet c_1, c_2, c_3 or c_4, c_5, c_6 of complex numbers. The property (6.6) is essential in the proof of existence and uniqueness of the solution and hence of the reflected and transmitted waves.

As a consequence of (5.3), the energy of the reflected and transmitted waves decays with distance from the interface.

7. Uniqueness in the reflection-transmission problem

Let $z < 0$ and $z > L$ be homogeneous, viscoelastic, half-spaces. The interval $0 < z < L$ consists of a multilayer of n adjacent inhomogeneous layers separated by interfaces at $z = z_1, ..., z_{n-1}$. At $z = z_0 = 0$ and $z = z_n = L$ an interface separates the multilayer from the adjacent half-space. The j-th layer, $j = 1, 2, ..., n$, occupies the interval $z \in (z_{j-1}, z_j)$. In each layer the constitutive properties depend continuously on the spatial coordinate z only. Let $f(z_-)$ and $f(z_+)$ denote the left and right-hand limits of a function f at z. As it is quite a common practice, we let the displacement $\mathbf{u}$ and the traction $\mathbf{t}$ be continuous across any interface. The continuity of $\mathbf{u}$ and $\mathbf{t}$ implies that

$$\mathbf{w}(z_{j-}) = \mathbf{w}(z_{j+}), \qquad j = 0, 1, ..., n.$$

To fix ideas we let the incident wave $\mathbf{w}^I$ come from $z = -\infty$. The wave $\mathbf{w}^I$ impinges on the multilayer at $z = 0$ and produces a reflected wave $\mathbf{w}^R$ at $z = 0$ and a transmitted wave at $z = L$. Since the half-spaces $z < 0$ and $z > L$ are homogeneous we denote by a superscript $-$ or $+$ the parameters pertaining to $z < 0$ or $z > L$. By (6.6), $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ represent forward waves, $\mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6$ represent backward waves. Consequently by (4.8) we can represent $\mathbf{w}^I, \mathbf{w}^R, \mathbf{w}^T$ as

$$\mathbf{w}^I(z) = \sum_{h=1}^{3} c_h^I \mathbf{p}_h^- \exp(i\sigma_h^- z), \qquad \mathbf{w}^R(z) = \sum_{h=4}^{6} c_h^R \mathbf{p}_h^- \exp(i\sigma_h^- z), \tag{7.1}$$

$$\mathbf{w}^T(z) = \sum_{h=1}^{3} c_h^T \mathbf{p}_h^+ \exp(i\sigma_h^+ (z - L)), \tag{7.2}$$

where, for formal convenience, we have chosen $z_0 = 0$ in the expressions of $\mathbf{w}^I, \mathbf{w}^R$ and $z_0 = L$ in the expression of $\mathbf{w}^T$. Also, let $\mathbf{w}^j$ be the solution to (4.2)

as $z \in (z_{j-1}, z_j)$. The whole (wave) solution is then expressed by

$$\mathbf{w}(z) = \begin{cases} \mathbf{w}^I(z) + \mathbf{w}^R(z), & z < 0, \\ \mathbf{w}^j(z), & z \in (z_{j-1}, z_j),\ j = 1, ..., n, \\ \mathbf{w}^T(z), & L < z. \end{cases} \tag{7.3}$$

The solution $\mathbf{w}(z)$ is also required to satisfy the boundary conditions, namely the continuity across the interfaces,

$$\begin{cases} \mathbf{w}^I(0_-) + \mathbf{w}^R(0_-) = \mathbf{w}^1(0_+), \\ \mathbf{w}^j(z_{j-}) = \mathbf{w}^{j+1}(z_{j+}), \quad j = 1, ..., n-1, \\ \mathbf{w}^n(L_-) = \mathbf{w}^T(L_+). \end{cases} \tag{7.4}$$

The RT problem reads: given $\mathbf{w}^I$, determine $\mathbf{w}^R$, $\mathbf{w}^T$, and $\mathbf{w}^j$, $j = 1, ..., n$, and hence $\mathbf{w}$ of (7.3), subject to (4.2) and (7.4).

We may phrase the problem by saying that the coefficients c_1^I, c_2^I, c_3^I are known and we have to determine the coefficients c_4^R, c_5^R, c_6^R, c_1^T, c_2^T, c_3^T and the functions $\mathbf{w}^j$, $j = 1, ..., n$, subject to (4.2) and (7.4).

THEOREM 2. *If $\mathbf{\Phi}^f$ is positive definite, as $z > L$, and $\mathbf{\Phi}^b$ is negative definite, as $z < 0$, then the solution (7.3) to the RT problem is unique.*

P r o o f. Let $\boldsymbol{\xi}, \boldsymbol{\chi}$ be two solutions to the RT problem, namely

$$\boldsymbol{\xi}(z) = \begin{cases} \mathbf{w}^I(z) + \boldsymbol{\xi}^R(z), & z < 0, \\ \boldsymbol{\xi}^j(z), & z \in (z_{j-1}, z_j),\ j = 1, ..., n, \\ \boldsymbol{\xi}^T(z), & L < z, \end{cases}$$

$$\boldsymbol{\chi}(z) = \begin{cases} \mathbf{w}^I(z) + \boldsymbol{\chi}^R(z), & z < 0, \\ \boldsymbol{\chi}^j(z), & z \in (z_{j-1}, z_j),\ j = 1, ..., n, \\ \boldsymbol{\chi}^T(z), & L < z. \end{cases}$$

When $z < 0$ and $z > L$, $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ can be represented as

$$\boldsymbol{\xi}^R(z) = \sum_{h=4}^{6} \xi_h^R \mathbf{p}_h^- \exp(i)\sigma_h^- z), \qquad \boldsymbol{\xi}^T(z) = \sum_{h=1}^{3} \xi_h^T \mathbf{p}_h^+ \exp(i\sigma_h^+(z-L)),$$

$$\boldsymbol{\chi}^R(z) = \sum_{h=4}^{6} \chi_h^R \mathbf{p}_h^- \exp(i\sigma_h^- z), \qquad \boldsymbol{\chi}^T(z) = \sum_{h=1}^{3} \chi_h^T \mathbf{p}_h^+ \exp(i\sigma_h^+(z-L)).$$

The difference $\mathbf{s} = \boldsymbol{\xi} - \boldsymbol{\chi}$ takes the form

$$\mathbf{s}(z) = \begin{cases} \mathbf{s}^R(z), & z < 0, \\ \mathbf{s}^j(z), & z \in (z_{j-1}, z_j), \quad j = 1, ..., n, \\ \mathbf{s}^T(z), & L < z, \end{cases}$$

and satisfies (4.2) and the continuity conditions

$$\begin{cases} \mathbf{s}^R(0_-) = \mathbf{s}^1(0_+), \\ \mathbf{s}^j(z_{j-}) = \mathbf{s}^{j+1}(z_{j+}), \quad j = 1, ..., n-1, \\ \mathbf{s}^n(L_-) = \mathbf{s}^T(L_+). \end{cases}$$

Moreover,

$$\mathbf{s}^R(0_-) = \sum_{h=4}^{6} s_h^R \mathbf{p}_h^-, \qquad \mathbf{s}^T(L_+) = \sum_{h=1}^{3} s_h^T \mathbf{p}_h^+,$$

where $s_h^R = \xi_h^R - \chi_h^R$, $s_h^T = \xi_h^T - \chi_h^T$.

By (6.6) we conclude that

$$\hat{\mathcal{F}}(\mathbf{s}(0_-)) \le 0, \qquad \hat{\mathcal{F}}(\mathbf{s}(L_+)) \ge 0. \tag{7.5}$$

Moreover, $\mathcal{F}(z) = \hat{\mathcal{F}}(\mathbf{s}(z))$ is continuous everywhere and satisfies (5.3) in the open intervals $(0, z_1)$, (z_1, z_2), ..., (z_{n-1}, L). Hence

$$\hat{\mathcal{F}}(\mathbf{s}(0_-)) \ge \hat{\mathcal{F}}(\mathbf{s}(L_+)). \tag{7.6}$$

It follows at once from (7.5) and (7.6) that

$$\hat{\mathcal{F}}(\mathbf{s}(0_-)) = 0, \qquad \hat{\mathcal{F}}(\mathbf{s}(L_+)) = 0.$$

Now, because $\mathbf{s}(0_-)$ is a linear combination of $\mathbf{p}_4^-, \mathbf{p}_5^-, \mathbf{p}_6^-$, the negative definiteness of $\mathbf{\Phi}^b$ yields

$$\hat{\mathcal{F}}(\mathbf{s}(0_-)) = 0 \quad \Longrightarrow \quad \mathbf{s}(0_-) = 0.$$

By continuity, also $\mathbf{s}(0_+) = 0$. The problem

$$\mathbf{s}' = \mathbf{A}\mathbf{s}, \qquad z \in (0, z_1), \qquad \mathbf{s}(0_+) = 0,$$

has the unique solution $\mathbf{s} = 0$. Hence $\mathbf{s}(z_{1-}) = 0$ and, by continuity $\mathbf{s}(z_{1+}) = 0$. Iteration of the argument leads to $\mathbf{s}(L_-) = 0$ and then $\mathbf{s}(L_+) = 0$. Hence $\mathbf{s}$ vanishes on $\mathbb{R}$ and the solution to the RT problem is unique. □

8. Existence in the reflection-transmission problem

To establish existence of the solution we proceed by showing how the solution can be determined. The continuity of $\mathbf{w}$ is used throughout. Within the j-th layer, $z \in (z_{j-1}, z_j)$, the solution $\mathbf{w}^j$ at z is given by $\mathbf{w}^j(z_{j-1})$ through the propagator $\mathbf{\Omega}^j$ at z, namely

$$\mathbf{w}^j(z) = \mathbf{\Omega}^j(z)\mathbf{w}^j(z_{j-1}), \qquad z \in (z_{j-1}, z_j).$$

Hence $\mathbf{\Omega}^j$ satisfies the differential equation

$$\frac{d}{dz}\mathbf{\Omega}^j = \mathbf{A}^j\mathbf{\Omega}^j, \qquad z \in (z_{j-1}, z_j), \qquad \mathbf{\Omega}^j(z_{j-1}) = \mathbf{1},$$

where $\mathbf{A}^j$ is the restriction of $\mathbf{A}$ to the j-th layer (z_{j-1}, z_j).

For brevity, let $\mathbf{\Omega}^j$ stand for $\mathbf{\Omega}^j(z_j)$ so that

$$\mathbf{w}(z_j) = \mathbf{\Omega}^j\mathbf{w}(z_{j-1}).$$

By the continuity of $\mathbf{w}$ across any interface we have

$$\mathbf{w}(L) = \mathbf{\Omega}\mathbf{w}(0), \tag{8.1}$$

where

$$\mathbf{\Omega} = \mathbf{\Omega}^n\mathbf{\Omega}^{n-1}\cdots\mathbf{\Omega}^1.$$

We represent the incident wave $\mathbf{w}^I$ and the reflected wave $\mathbf{w}^R$, at $z < 0$, and the transmitted wave $\mathbf{w}^T$, at $z > L$, in the form (7.1). To solve the RT problem we need to express $\{c_1^T, c_2^T, c_3^T, c_4^R, c_5^R, c_6^R\}$ in terms of $\{c_1^I, c_2^I, c_3^I\}$. By (8.1) we have

$$\mathbf{\Omega}[\mathbf{w}^I(0) + \mathbf{w}^R(0)] = \mathbf{w}^T(L)$$

whence, by means of the continuity at $z = 0, L$,

$$\sum_{h=1}^{3} c_h^T\mathbf{p}_h^+ - \sum_{h=4}^{6} c_h^R\mathbf{\Omega}\mathbf{p}_h^- = \sum_{h=1}^{3} c_h^I\mathbf{\Omega}\mathbf{p}_h^- \tag{8.2}$$

where $\mathbf{p}^+$ is evaluated at L_+ and $\mathbf{p}^-$ at 0_-. This is the system to be solved for the RT problem in a layer.

THEOREM 3. *The vectors $\mathbf{p}_1^+, \mathbf{p}_2^+, \mathbf{p}_3^+$ and $\mathbf{\Omega}\mathbf{p}_4^-, \mathbf{\Omega}\mathbf{p}_5^-, \mathbf{\Omega}\mathbf{p}_6^-$ are linearly independent, which provides the existence of the solution to the RT problem.*

P r o o f. Starting from

$$\alpha_1\mathbf{p}_1^+ + \alpha_2\mathbf{p}_2^+ + \alpha_3\mathbf{p}_3^+ + \alpha_4\mathbf{\Omega}\mathbf{p}_4^- + \alpha_5\mathbf{\Omega}\mathbf{p}_5^- + \alpha_6\mathbf{\Omega}\mathbf{p}_6^- = 0$$

we have

$$\hat{\mathcal{F}}(-\alpha_1\mathbf{p}_1^+ - \alpha_2\mathbf{p}_2^+ - \alpha_3\mathbf{p}_3^+) = \hat{\mathcal{F}}(\mathbf{\Omega}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-)). \tag{8.3}$$

Consider the field $\mathbf{w}(z)$, such that

$$\mathbf{w}(0) = (\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-), \qquad \mathbf{w}(L) = \mathbf{\Omega}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-),$$

and observe that

$$\mathcal{F}(0_+) = \hat{\mathcal{F}}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-), \qquad \mathcal{F}(L_-) = \hat{\mathcal{F}}(\mathbf{\Omega}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-)).$$

By (5.3) we have $\mathcal{F}(0_+) \geq \mathcal{F}(L_-)$ and hence

$$\hat{\mathcal{F}}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-) = \mathcal{F}(0_+) \geq \mathcal{F}(L_-) = \hat{\mathcal{F}}(\mathbf{\Omega}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-)). \tag{8.4}$$

By (8.3), (8.4), and (6.6) we have

$$0 \leq \sum_{h,k=1}^{3} \alpha_h^* \alpha_k \Phi_{hk}^f = \hat{\mathcal{F}}(-\alpha_1\mathbf{p}_1^+ - \alpha_2\mathbf{p}_2^+ - \alpha_3\mathbf{p}_3^+)$$

$$\leq \hat{\mathcal{F}}(\alpha_4\mathbf{p}_4^- + \alpha_5\mathbf{p}_5^- + \alpha_6\mathbf{p}_6^-) = \sum_{h,k=4}^{6} \alpha_h^* \alpha_k \Phi_{hk}^b \leq 0.$$

Hence we have

$$\sum_{h,k=1}^{3} \alpha_h^* \alpha_k \Phi_{hk}^f = 0, \qquad \sum_{h,k=4}^{6} \alpha_h^* \alpha_k \Phi_{hk}^b = 0.$$

The definiteness of the two blocks of $\mathbf{\Phi}$ implies that $\alpha_1, \alpha_2, ..., \alpha_6 = 0$.

The linear independence of the vectors $\mathbf{p}_1^+, \mathbf{p}_2^+, \mathbf{p}_3^+$ and $\mathbf{\Omega}\mathbf{p}_4^-, \mathbf{\Omega}\mathbf{p}_5^-, \mathbf{\Omega}\mathbf{p}_6^-$ makes the system (8.2) to have a (unique) solution. □

REMARK 2. The particular case $\mathbf{\Omega} = \mathbf{1}$ describes the RT problem at the interface between two half-spaces. In such a case existence and uniqueness hold with $\mathbf{k}_\perp \in \mathbb{C}^3$. Moreover, existence and uniqueness hold for a free surface and for a fixed surface, bounding a half-space.

9. Block structure of $\mathbf{\Phi}$

We now ascertain that, in two significant circumstances, the matrix $\mathbf{\Phi}$ has the block structure (6.5) with the positive-negative definiteness. Following [9], we observe that if

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a} \exp[i(\mathbf{k}_\perp \cdot \mathbf{x} + \sigma z + \omega t)]$$

then the propagation condition takes the form (4.5). The eigenvectors $\{\mathbf{p}_\alpha\}$ satisfy (4.11) whence

$$\mathbf{p}_\beta^T \mathbf{K} \mathbf{p}_\alpha = \mathbf{a}_\beta \cdot \mathbf{l}_\alpha + \mathbf{l}_\beta \cdot \mathbf{a}_\alpha = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 2\mathbf{a}_\alpha \cdot \mathbf{l}_\alpha =: \gamma_\alpha & \text{if } \alpha = \beta. \end{cases} \tag{9.1}$$

9.1. Oblique incidence of homogeneous waves in elastic half-spaces

The matrix $\mathbf{G}$ and the wave vector $\mathbf{k}_\perp$ are real-valued. Hence by (4.5) it follows that the roots σ_α are real or occur in complex-conjugate pairs. Restrict our attention to real-valued roots.

Since σ_α is real then $\mathbf{a}_\alpha$ is real and, by (4.7), $\mathbf{l}_\alpha$ is imaginary. Consequently,

$$\sigma_\alpha \in \mathbb{R} \quad \Longrightarrow \quad \mathbf{a}_\alpha = \mathbf{a}_\alpha^*, \quad \mathbf{l}_\alpha = -\mathbf{l}_\alpha^*.$$

Hence, by (6.2) and (9.1),

$$\Phi_{\alpha\beta} = i\frac{\omega}{4}(\mathbf{a}_\alpha \cdot \mathbf{l}_\beta + \mathbf{l}_\alpha \cdot \mathbf{a}_\beta) = i\frac{\omega}{4}\mathbf{p}_\alpha^T \mathbf{K}\mathbf{p}_\beta = i\frac{\omega}{4}\gamma_\alpha\delta_{\alpha\beta}.$$

This means that, in the α-th column, only the diagonal term is nonzero and its value is real.

The whole matrix $\mathbf{\Phi}$ is diagonal and, by (6.4), three eigenvalues are positive and three are negative. Consequently, possibly by reordering the terms, the form (6.5) holds with $\tilde{\mathbf{\Phi}} = 0$ and $\mathbf{\Phi}^f, \mathbf{\Phi}^b$ diagonal.

9.2. Normal incidence in viscoelastic isotropic half-spaces

Since $\mathbf{k}_\perp = 0$ then $\mathbf{A}$ takes the block form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & (\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1} \\ -\rho\omega^2\mathbf{1} & \mathbf{0} \end{bmatrix}.$$

The isotropy of the solid results in

$$\mathbf{e}_3\mathbf{G}\mathbf{e}_3 = \mu\mathbf{1} + (\mu+\lambda)\mathbf{e_3} \otimes \mathbf{e_3},$$

where $\mu, \lambda \in \mathbb{C}$. The thermodynamic restriction (2.5) provides the inequalities

$$\omega\Im\mu > 0, \quad \omega\Im(2\mu+\lambda) > 0, \quad \omega \in \mathbb{R} \setminus \{0\}.$$

Hence

$$(\mathbf{e}_3\mathbf{G}\mathbf{e}_3)^{-1} = \frac{1}{\mu}\mathbf{1} - \frac{\mu+\lambda}{\mu(\mathbf{2}\mu+\lambda)}\mathbf{e_3} \otimes \mathbf{e_3}$$

and $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2\mu+\lambda) \\ -\rho\omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho\omega^2 & 0 & 0 & 0 \end{bmatrix}.$$

Because $1/\mu$ and $1/(2\mu+\lambda)$ are in the fourth quadrant, as $\omega > 0$, we let

$$\sigma_T = \sqrt{\rho\omega^2/\mu}, \qquad \sigma_L = \sqrt{\rho\omega^2/(2\mu+\lambda)},$$

and say that σ_L, σ_T are in the second quadrant as $\omega > 0$, in the first one as $\omega < 0$. Once the eigenvectors of $\mathbf{A}$ are determined we can write the matrix $\mathbf{P}$ in the form

$$\mathbf{P} = \begin{bmatrix} 0 & -\sigma_T & 0 & 0 & \sigma_T & 0 \\ 0 & 0 & -\sigma_T & 0 & 0 & \sigma_T \\ \sigma_L & 0 & 0 & -\sigma_L & 0 & 0 \\ 0 & -i\rho\omega^2 & 0 & 0 & -i\rho\omega^2 & 0 \\ 0 & 0 & -i\rho\omega^2 & 0 & 0 & -i\rho\omega^2 \\ i\rho\omega^2 & 0 & 0 & i\rho\omega^2 & 0 & 0 \end{bmatrix}.$$

A direct application of (6.1) yields

$$\mathbf{\Phi} = -\frac{\rho\omega^3}{4} \begin{bmatrix} \sigma_L+\sigma_L^* & 0 & 0 & -(\sigma_L-\sigma_L^*) & 0 & 0 \\ 0 & \sigma_T+\sigma_T^* & 0 & 0 & -(\sigma_T-\sigma_T^*) & 0 \\ 0 & 0 & \sigma_T+\sigma_T^* & 0 & 0 & -(\sigma_T-\sigma_T^*) \\ \sigma_L-\sigma_L^* & 0 & 0 & -(\sigma_L+\sigma_L^*) & 0 & 0 \\ 0 & \sigma_T-\sigma_T^* & 0 & 0 & -(\sigma_T+\sigma_T^*) & 0 \\ 0 & 0 & \sigma_T-\sigma_T^* & 0 & 0 & -(\sigma_T+\sigma_T^*) \end{bmatrix}.$$

Because $\omega(\sigma_L+\sigma_L^*) < 0$ and $\omega(\sigma_T+\sigma_T^*) < 0$ for any $\omega \neq 0$ it follows that the block $\mathbf{\Phi}^f$ of $\mathbf{\Phi}$ is positive definite and the block $\mathbf{\Phi}^b$ is negative definite.

10. Inhomogeneous waves and nondefiniteness in elastic solids

In elastic solids the values σ_α are real or pairwise complex conjugate. Letting σ_h and σ_{h+3} be complex conjugate, $\sigma_{h+3} = \sigma_h^* \neq \sigma_h$, we have

$$\mathbf{a}_{h+3} = \mathbf{a}_h^*, \qquad \mathbf{l}_{h+3} = -\mathbf{l}_h^*. \tag{10.1}$$

By means of the orthogonality condition (4.11) we now examine the form of $\mathbf{\Phi}$. By (10.1) and (4.11) we have

$$\Phi_{h,h+3} = i\frac{\omega}{4}(\mathbf{a}_h^* \cdot \mathbf{l}_{h+3} - \mathbf{l}_h^* \cdot \mathbf{a}_{h+3}) = i\frac{\omega}{4}(\mathbf{a}_{h+3} \cdot \mathbf{l}_{h+3} + \mathbf{l}_{h+3} \cdot \mathbf{a}_{h+3}) = i\frac{\omega}{4}\gamma_{h+3}.$$

For any value of β, by (10.1) we have

$$\Phi_{h\beta} = i\frac{\omega}{4}(\mathbf{a}_h^* \cdot \mathbf{l}_\beta - \mathbf{l}_h^* \cdot \mathbf{a}_\beta) = i\frac{\omega}{4}(\mathbf{a}_{h+3} \cdot \mathbf{l}_\beta + \mathbf{l}_{h+3} \cdot \mathbf{a}_\beta).$$

By (4.11), the condition $\beta \neq h+3$ gives $\Phi_{h\beta} = 0$. The result

$$\Phi_{h\beta} = \begin{cases} i\omega\gamma_{h+3}/4, & \beta = h+3 \\ 0, & \beta \neq h+3 \end{cases}$$

holds as a consequence of the conjugacy property $\sigma_{h+3} = \sigma_h^*$, irrespective of the fact that the other values of σ_α are real or complex. On the other hand, for real values of σ_α, only the corresponding diagonal entry, $\Phi_{\alpha\alpha}$ is nonzero. Hence for any row and any column, only one entry is nonzero. As an example, if $\sigma_1, \sigma_4 \in \mathbb{R}$ and $\sigma_5 = \sigma_2^*$, $\sigma_6 = \sigma_3^*$, then

$$\mathbf{\Phi} = i\frac{\omega}{4}\begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6 \\ 0 & 0 & 0 & \gamma_4 & 0 & 0 \\ 0 & -\gamma_5^* & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_6^* & 0 & 0 & 0 \end{bmatrix}.$$

Of course the nonzero entries of $\mathbf{\Phi}$ depends on the eigenvectors $\mathbf{p}_\alpha$ of $\mathbf{A}$.

It is an immediate consequence of this evaluation that the occurrence of complex values of σ_α makes the diagonal blocks of $\mathbf{\Phi}$ non-definite. Since the definiteness of the blocks is essential for the uniqueness of the solution, we can say that the uniqueness of the solution fails if some values of σ_α are complex.

Complex values of σ_α denote that the corresponding solutions occur in the form

$$\mathbf{w}_\alpha \exp[i(\mathbf{k}_\perp \cdot \mathbf{x} + \omega t)] = \mathbf{p}_\alpha \exp(-\Im\sigma_\alpha\, z) \exp[i(\mathbf{k}_\perp \cdot \mathbf{x} + \Re\sigma_\alpha\, z + \omega t)]$$

namely as inhomogeneous (or evanescent) waves. Hence nonuniqueness is associated with the occurrence of inhomogeneous waves which may be viewed as localized waves. This gives an interesting interpretation of the result, namely the occurrence of localized waves implies nonuniqueness in the RT problem. This conclusion is consistent with the nonuniqueness associated with the occurrence of interface waves.

11. Remarks about nonuniqueness

Nonexistence and/or nonuniqueness may look not very convincing from the physical point of view. Here we comment on the lack of uniqueness in elasticity.

In this sense it is instructive to consider waves in elastic solids and let two values of σ be complex-conjugate.

To fix ideas, let $\sigma_1, \sigma_2, \sigma_4, \sigma_5 \in \mathbb{R}$ and $\sigma_3, \sigma_6 \in \mathbb{C}$, $\sigma_3 = \sigma_6^*$. Hence

$$\mathbf{\Phi} = i\frac{\omega}{4}\begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6 \\ 0 & 0 & 0 & \gamma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & -\gamma_6^* & 0 & 0 & 0 \end{bmatrix}.$$

Restrict attention to $z = z_0$. By (6.3) it follows that

$$\hat{\mathcal{F}}(\mathbf{p}_3) = 0, \qquad \hat{\mathcal{F}}(\mathbf{p}_6) = 0$$

and

$$\hat{\mathcal{F}}(\mathbf{p}_3 + a\mathbf{p}_6) = i\frac{\omega}{4}(a\gamma_6 - a^*\gamma_6^*).$$

Letting $a = \pm 1$ we have the values

$$\hat{\mathcal{F}}(\mathbf{p}_3 \pm \mathbf{p}_6) = \pm i\frac{\omega}{4}(\gamma_6 - \gamma_6^*)$$

of the energy flux. To fix ideas let $i\omega(\gamma_6 - \gamma_6^*) > 0$. This means that the waves associated with $\mathbf{p}_3$ and $\mathbf{p}_6$ are neither forward nor backward waves and nevertheless, the superpositions $\mathbf{p}_3 + \mathbf{p}_6$ and $\mathbf{p}_3 - \mathbf{p}_6$ are forward and backward, respectively.

In the basis $\tilde{\mathbf{p}}_1 = \mathbf{p}_1, \tilde{\mathbf{p}}_2 = \mathbf{p}_2, \tilde{\mathbf{p}}_3 = \mathbf{p}_3 + \mathbf{p}_6$ for the forward waves and $\tilde{\mathbf{p}}_4 = \mathbf{p}_4, \tilde{\mathbf{p}}_5 = \mathbf{p}_5, \tilde{\mathbf{p}}_6 = \mathbf{p}_3 - \mathbf{p}_6$ for the backward vaves, the first 3×3 diagonal block of $\tilde{\mathbf{\Phi}} = \tilde{\mathbf{P}}^\dagger \boldsymbol{\mathcal{I}} \tilde{\mathbf{P}}$ is positive definite and the second one is negative definite. The positive-negative definiteness of the diagonal blocks then implies that, in the basis $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 + \mathbf{p}_6$ for the forward waves and $\mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_3 - \mathbf{p}_6$ for the backward vaves, the solution to the RT problem exists and is unique.

However, for any $a \in \mathbb{R}$, the triplets $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 + |a|\mathbf{p}_6$ and $\mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_3 - |a|\mathbf{p}_6$ can be bases for the forward and backward waves. The solution then exists and is unique but depends on the parameter a. Consequently, the solution to the RT problem depend on the basis chosen. This shows that the nonuniqueness of the solution, in elasticity, occurs because complex values of σ arise (inhomogeneous waves). The crucial property for the nonuniqueness is vanishing of $\mathcal{F}$ for inhomogeneous waves in elasticity. In viscoelasticity, instead, inhomogeneous waves usually provide $\mathcal{F} \neq 0$ and hence uniqueness is allowed (see Sec. 9.2).

The amplitude of the wave

$$\mathbf{p}_3 \exp(-\Im\sigma z) \exp[i(\mathbf{k}_\perp \cdot \mathbf{x} + \Re\sigma z + \omega t)], \quad \Im\sigma > 0,$$

decays as z increases. Sometimes [20, 9] the decay is regarded as a characterization of forward propagation. By arguing in this way one may say that a transmitted wave can be represented as a superposition of the waves associated with $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. By requiring that the continuity conditions hold at the interfaces, we obtain an algebraic system for the unknown amplitudes of the waves. Once the choice of outgoing waves (possibly partly with the energy criterion and partly with the amplitude criterion) is made, the RT problem is uniquely solved provided only the algebraic system allows for a unique solution (algebraic uniqueness). This means that the solution is unique once the basis is selected. It is a matter of fact that also such a basis may be associated with incompatibilities of the algebraic system (see [9]).

12. Conclusions

The existence and uniqueness results obtained in this paper hold for time-harmonic waves in a stratified anisotropic viscoelastic solid whose geometry is that of a layer sandwiched between homogeneous half-spaces. The oblique character of the solution is modelled through the factor $\exp(i\mathbf{k}_\perp \cdot \mathbf{x})$. Hence the vector $[\mathbf{u}_F, \mathbf{t}_F]$ is found to satisfy the first-order system (4.2). The matrix $\mathbf{A}$ is required to be simple and to have non-zero eigenvalues. If $\mathbf{k}_\perp \in \mathbb{R}^3$ then we have the decay property $\mathcal{F}' \leq 0$ for the energy flux $\mathcal{F}$. This property, along with the partition of elementary waves, is the conceptual ingredient for existence and uniqueness of the solution in multilayers between half-spaces, in bonded half-spaces and in a free half-space.

Existence and uniqueness are not shown to hold when the diagonal blocks $\mathbf{\Phi}^f$ and $\mathbf{\Phi}^b$ of $\mathbf{\Phi}$ are not definite. Such is the case if (the solid is elastic and) the fundamental solutions comprise evanescent waves. This is not surprising because the occurrence of evanescent waves means that the number of energy-carrying waves is not great enough to represent the reflected and transmitted waves.

The characterization of the direction of propagation of the fundamental solutions, by means of the energy flux $\mathcal{F}$, is not new in the literature [20, 9]. The main result of this paper is that $\mathcal{F}$ obeys the decay property, $\mathcal{F}' \leq 0$, which in turn yields existence and uniqueness of the solution to the RT problem, in dissipative solids, if the diagonal blocks of $\mathbf{\Phi}$ satisfy the positive-negative definiteness.

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