

Direct methods for limits in plasticity

A. R. S. PONTER, H. F. CHEN

*Mathematical Modelling Centre,
Department of Engineering, University of Leicester,
Leicester, LE1 7RH, UK
e-mail: asp@le.ac.uk*

THE PAPER discusses a methodology for the evaluation of shakedown and ratchet limits for an elastic perfectly plastic solid subjected to mechanical and thermal cycles of loading. The steady cyclic state is characterised by a minimum theorem that contains the classical shakedown theorems as a special case. For a prescribed class of kinematically admissible strain rate histories, the minimum of the functional is found by a programming method, the Linear Matching Method, which converges to the least upper bound. Three examples are given for a finite element implementation, rolling contact on a half-space, the behaviour of a complex heat exchanger and the behaviour of a regular particulate metal matrix composite subjected to variable temperature.

1. Introduction

SINCE HUBER'S landmark discovery of the yield condition, now known as the Huber–von Mises condition, plasticity theory has developed as a methodology capable of providing profound insights into material processing and structural design. Where design is concerned, the most significant insight was provided by the classical theorems of limit load and shakedown limit analysis in the 1950's [1]. Structural design codes currently in use call upon this insight in guarding against excessive deformation. At the same time computer methods have developed along two parallel paths, the development of complex constitutive relationships for structural simulation and the development of so-called direct methods. Direct methods refer to theoretical and computational methods that directly address the quantities required in a design situation, e.g. the factor of safety against a design limiting condition. The most widely discussed method consists in the evaluation of the shakedown limit, providing a limit against excessive deformation. The range of design restrictions required in practice, however, is much wider. These including, for example, fatigue limits, creep deformation limits, creep rupture limits and creep/fatigue limits for structures that suffer severe mechanical and thermal loading. There has been a developing need

to produce direct methods with a wider range of theoretical background than shakedown theory. Such methods should have the potential for producing computer methods that call upon the strengths of classical plasticity theory but with application to a wide range of material behaviour and structural circumstances.

This paper describes a possible way forward in such an endeavour. Structural design usually calls upon an understanding of performance for a repeated cycle of loading. Power stations perform cycles of operation, interspersed with maintenance periods; turbines likewise are cyclically loaded, e.g. a single flight of an aircraft. Design codes, implicitly, look at a typical cycle within a steady state. Hence it is useful to first characterise cyclic behaviour as a minimum theorem for a class of kinematically admissible inelastic strain rate histories or equilibrium stress histories. The location of the minimum reduces to a programming problem, and those familiar with shakedown analysis will be conscious of the number of ways that this may be achieved. Recently, however, a particular type of method has emerged that has the advantage of remaining close to conventional structural analysis and may be implemented within standard commercial finite element codes. The method originates from methods of approximate structural analysis by MARRIOT (Reduced modulus method) [2], SHESHADRI (R-node method) [3] and MACKENZIE and BOYLE (Elastic Compensation Method) [4]. The essential concept is to describe non-linear inelastic material behaviour by linear solutions where the material coefficients vary spatially and in time. This provides a type of functional representation that is particularly flexible. It is possible to develop the idea into properly convergent numerical methods capable of application to minimum theorems that characterise steady cycle state behaviour. This approach has been termed the Linear Matching Method by the authors of this paper.

In this paper we describe the application of this approach to an elastic-perfectly plastic material subjected to cyclic loading. With extensions to high temperature creep the method has been applied to all the stages of the UK's life assessment method, R5, [6, 7] for high temperature power plant, providing economic and accurate limits, capable of extending the range and accuracy of present methods [5]. Here we confine ourselves to problems where creep is not an issue and give examples from three contrasting areas of application, shakedown under rolling contact, the behaviour of a power plant heat exchanger, and the behaviour of a metal matrix composite when subjected to constant macro-stress and variable temperature.

In the following sections we briefly summarise a general cyclic minimum theorem for perfect plasticity and the application of the Linear Matching Method for a particular class of problems. This is followed by a description of the three examples.

2. Cyclic behaviour

Consider a body with volume V and surface S . Part of S , namely S_T , is subjected to a cyclic history of load $\lambda P_i(x_i, t)$, while the remainder of S , namely S_U , is subjected to zero displacement rate, $\dot{u}_i = 0$. Within V a cyclic history of temperature $\lambda\theta(x_i, t)$ occurs. Here λ denotes a load parameter. The body is composed of an elastic/perfectly plastic solid. For such a model the modes of behaviour are well understood. For differing loading conditions, four modes of behaviour are possible.

Elastic behaviour: For sufficiently low values of λ , the elastic stress history $\lambda\hat{\sigma}_{ij}$ due to λP_i and $\lambda\theta$ lies within yield, assuming no initial residual stresses.

Shakedown: When the elastic stress history exceeds yield, plastic strains occur during initial cycles and the stress history σ_{ij} asymptotes to a history of the form $\sigma_{ij} = \lambda\hat{\sigma}_{ij}(t) + \bar{\rho}_{ij}$, where $\hat{\sigma}_{ij}$ denotes the linear elastic stress history for $\lambda = 1$ and $\bar{\rho}_{ij}$ denotes a time-constant residual stress field. Generally $\bar{\rho}_{ij}$ is not unique and depends on any initial residual stress field but the value of $\lambda = \lambda_s$ that characterises the limit to this mode of behaviour, the shakedown limit, is independent of any initial residual stress. At the limit a unique $\bar{\rho}_{ij}$ exists for a strictly convex yield surface. The shakedown limit may be subdivided into two subcategories, a reverse plasticity limit (low P_i , high θ) where a closed cycle of plastic strain begins to occur but no cyclic strain growth, and a ratchet limit (high P_i , low θ) where cyclic strain growth occurs at values of $\lambda = \lambda_s < \lambda_L$ where λ_L is the limit load value for constant temperature and maximum load.

In excess of the shakedown limit, the steady state cyclic stress history has the form $\sigma_{ij} = \lambda\hat{\sigma}_{ij} + \bar{\rho}_{ij} + \rho_{ij}^r$ where $\bar{\rho}_{ij}$ now becomes the residual stress at the beginning and the end of the cycle and ρ_{ij}^r denote the change in the residual stress field during the cycle, reducing to zero at the beginning and end of the cycle. This stress history is produced by a cyclic history of plastic strain $\hat{\epsilon}_{ij}^p$ that accumulates over the cycle to a compatible increment of strain $\Delta\epsilon_{ij}^p$, giving rise to an increment of displacement Δu_i^p . In this state two possible modes of behaviour occur, corresponding to two separate regions of loading.

Plastic shakedown or reverse plasticity: For this range of loading, in excess of the reverse plasticity shakedown limit, no cyclic strain growth occurs, i.e. $\Delta\epsilon_{ij}^p = 0$ and $\Delta u_i^p = 0$, but locally a closed cycle of plastic strain occurs as a potential source of fatigue initiation. Hence the interest in this region is an evaluation of the amplitude and location of the maximum plastic strain amplitude.

Plastic ratchetting: For this range $\Delta\epsilon_{ij}^p$ and Δu_i^p are non-zero and the material experiences a cyclic growth of strain.

The evaluation of these patterns of behaviour may be obtained through a large number of step-by-step finite element calculations and this has certainly been done in the past. The identification of the boundaries between these behav-

ioural load regions can be difficult and essentially subjective. The whole calculation is rather excessively tedious. However, we show, in the next two sections, how it is possible to identify the primary characteristics of the region boundaries directly. These methods have been developed to provide improved computational methods for structural life assessment methods for high temperature power plant [5, 8, 9, 15, 16].

In Sec. 3 the minimum theorem is described and related to the classical shakedown theorems. The Linear Matching Method is summarised for shakedown in Sec. 4 and the solution strategy described. This is extended to the ratchet limit in Sec. 5. Finally, in Sec. 6 numerical solutions are discussed.

3. Minimum theorems

The theory discussed in this section is derived in general terms in PONTER and CHEN [8] and CHEN and PONTER [9]. Since the publication of these papers it has become clear that the theorems were contained, in essence, in the classic book of GOKHFELD and CHERIAVSKY [19]. Recently POLIZZOTTO has extended the result to a generalized standard material [20]. Here the results are summarised for perfect plasticity. The yield condition is given by the Huber–von Mises condition and the associated flow rule,

$$(3.1) \quad f(\sigma_{ij}) = \bar{\sigma} - \sigma_y \leq 0, \quad \dot{\varepsilon}_{ij}^p = \dot{\mu} \frac{\partial f}{\partial \sigma_{ij}},$$

where $\bar{\sigma}$ denotes the von Mises effective stress and σ_y the uniaxial yield stress.

The linear elastic stress solution, corresponding to plastic strains $\dot{\varepsilon}_{ij}^P = 0$, is denoted by $\lambda \hat{\sigma}_{ij}$, with

$$(3.2) \quad \lambda \hat{\sigma}_{ij}(x, t) = \lambda(\hat{\sigma}_{ij}^P(x, t) + \hat{\sigma}_{ij}^\theta(x, t)),$$

where $\hat{\sigma}_{ij}^P(x, t)$ and $\hat{\sigma}_{ij}^\theta(x, t)$ are the linear elastic stress solutions corresponding to P_i and θ .

For the above general problem in a typical cycle, $0 \leq t \leq \Delta t$, in the steady state, the following minimum theorem (PONTER and CHEN [8]) exists. For any chosen value of λ , the functional

$$(3.3) \quad I(\dot{\varepsilon}_{ij}^c, \lambda) = \int_V \int_0^{\Delta t} (\sigma_{ij}^c - \lambda \hat{\sigma}_{ij}) \dot{\varepsilon}_{ij}^c dt dV$$

is minimised by the exact solution, where σ_{ij}^c denotes the stress at yield corresponding to a plastic strain rate history $\dot{\varepsilon}_{ij}^p = \dot{\varepsilon}_{ij}^c$, such that the accumulated strain over the cycle,

$$(3.4) \quad \int_0^{\Delta t} \dot{\varepsilon}_{ij}^c = \Delta \varepsilon_{ij}^c$$

is kinematically admissible, i.e. compatible with a displacement field, Δu_{ij}^c , which, in turn, satisfies the displacement boundary condition on S_u . Two additional restrictions are now placed on the magnitude of $\dot{\varepsilon}_{ij}^c$:

RESTRICTION 1: Corresponding to $\dot{\varepsilon}_{ij}^c$, a cyclic history of residual stress, $\rho_{ij}^c(x, t)$, is defined such that it satisfies the relationship

$$(3.5) \quad \dot{\varepsilon}_{ij}^{cc} = C_{ijkl} \dot{\rho}_{ij}^c + \dot{\varepsilon}_{ij}^c,$$

where $\dot{\varepsilon}_{ij}^{cc}$ is kinematically admissible and C_{ijkl} denotes the linear elastic compliance tensor. Note that:

$$(3.6) \quad \rho_{ij}^c(0) = \rho_{ij}^c(\Delta t) = 0.$$

RESTRICTION 2: Corresponding to $\rho_{ij}^c(x, t)$, a restriction is then placed on the absolute magnitude of $\dot{\varepsilon}_{ij}^c$, with the requirement that there exists a constant residual stress field, $\bar{\rho}_{ij}$, such that the composite stress history

$$(3.7) \quad \sigma_{ij} = \hat{\sigma}_{ij} + \bar{\rho}_{ij} + \rho_{ij}^c$$

satisfies the yield condition, $f(\sigma_{ij}) \leq 0$, for $0 \leq t \leq \Delta t$.

For a prescribed load history, i.e. a prescribed λP_i and $\lambda \theta$,

$$(3.8) \quad I(\dot{\varepsilon}_{ij}^c, \lambda) \geq 0$$

with equality achieved when $\dot{\varepsilon}_{ij}^c = \dot{\varepsilon}_{ij}^{cr}$, the exact cyclic solution, PONTÉ and CHEN [8].

This result includes the classical shakedown theorems as a limiting case. If we take $\lambda = \lambda_s$, the value corresponding to the shakedown limit, the magnitude of the strain rate history, $\dot{\varepsilon}_{ij}^c$, becomes infinitesimally small and $\rho_{ij}^c(x, t)$ becomes insignificant compared with the elastic stresses. Restriction 1 no longer applies and Restriction 2 corresponds to the lower bound shakedown theorem (KOITER [1]). At this limit I is linear in the absolute magnitude of $\dot{\varepsilon}_{ij}^c$. Hence (2.8) yields that $I(\dot{\varepsilon}_{ij}^c, \lambda_s) \geq 0$ and $I(\dot{\varepsilon}_{ij}^s, \lambda_s) = 0$, where $\dot{\varepsilon}_{ij}^s$ is the exact shakedown mechanism and we understand that $\dot{\varepsilon}_{ij}^c$ and $\dot{\varepsilon}_{ij}^s$ are small but finite. If we now define $\lambda = \lambda_{UB}$ as the value that satisfies $I(\dot{\varepsilon}_{ij}^c, \lambda_{UB}^c) = 0$, independent of the absolute magnitude of $\dot{\varepsilon}_{ij}^c$, then it follows, easily, that $\lambda_{UB}^c \geq \lambda_s$. Hence the upper bound shakedown theorem of KOITER [1] is recovered. In summary, for small $\dot{\varepsilon}_{ij}^c$, the aforesaid minimum theorem provided a generalisation of both the lower and upper bound shakedown theorems. For finite $\dot{\varepsilon}_{ij}^c$ the theorem is applicable above the shakedown limit.

4. The linear matching method for the shakedown limit

The Linear Matching Method involves the solution of a sequence of linear problems that yield kinematically admissible strain rate histories corresponding to monotonically reducing upper bounds λ_{UB} that may be shown to converge to the least upper bound associated with the class of displacement fields chosen, PONTER and ENGELHARDT [10]. The linear material is matched to the yield condition for the current strain rate history. Essentially it is a non-linear programming method but of a form that is particularly easy to understand and implement.

For simplicity, consider a problem where the elastic solution has two extremes at times t_1 and t_2 with linear interpolation between these values at other times. In this case the plastic strain history consists of two increments of strain $\Delta\varepsilon_{ij}^1$ and $\Delta\varepsilon_{ij}^2$, corresponding to the extremes of the elastic stress history at times t_1 and t_2 . For initial estimates $\Delta\varepsilon_{ij}^{1I}$ and $\Delta\varepsilon_{ij}^{2I}$, we define shear moduli $\bar{\mu}^{1I}$ and $\bar{\mu}^{2I}$ corresponding to an incompressible, isotropic linear material by the matching conditions;

$$(4.1) \quad \sigma_y = \left(\frac{3}{2}\right) 2\bar{\mu}^{lI} \bar{\varepsilon}(\Delta\varepsilon_{ij}^{lI}), \quad l = 1, 2,$$

where $\bar{\varepsilon}$ denotes the von Mises effective strain. For an incompressible linear materials defined by shear moduli $\bar{\mu}^{1I}$ and $\bar{\mu}^{2I}$, the effective stress matches the yield condition at these, initial, estimates of plastic strain increments. A new estimate of the strain rate history is then given by the solution of the following linear problem:

$$(4.2) \quad \begin{aligned} \Delta\varepsilon_{ij}^{1F} &= \frac{1}{2\bar{\mu}^{1I}} \{ \lambda \hat{\sigma}_{ij}(t_1) + \bar{\rho}_{ij}^F \}, \\ \Delta\varepsilon_{ij}^{2F} &= \frac{1}{2\bar{\mu}^{2I}} \{ \lambda \hat{\sigma}_{ij}(t_2) + \bar{\rho}_{ij}^F \}, \\ \lambda &= \lambda_{UB}^I; \end{aligned}$$

$$(4.3) \quad \begin{aligned} \Delta\varepsilon_{ij}^F &= \Delta\varepsilon_{ij}^{1F} + \Delta\varepsilon_{ij}^{2F}, \quad \hat{\sigma}_{ij}(t) = (\hat{\sigma}_{ij}^p + \hat{\sigma}_{ij}^\theta), \\ \Delta\varepsilon_{kk}^{1F} &= 0, \\ \Delta\varepsilon_{kk}^{2F} &= 0, \end{aligned}$$

where $\Delta\varepsilon_{ij}^F$ is kinematically admissible and $\bar{\rho}_{ij}^F$ satisfies equilibrium. Here λ_{UB}^I denotes the upper bound corresponding to the initial solution, i.e. $I(\Delta\varepsilon_{ij}^{lI}, \lambda_{UB}^I) = 0$. General theory (PONTER and ENGELHARDT [10]) then gives that

$$(4.4) \quad \lambda_{UB}^F \leq \lambda_{UB}^I$$

with equality at convergence. Repeated application of this algorithm produces a sequence of reducing upper bounds that converge to the least upper bound corresponding to the finite element mesh for finite element solution. The generalisation to an arbitrary strain rate history is simple and is given in [10].

5. Evaluation of the ratchet limit

In the case of loading in excess of shakedown, a parallel understanding of the nature of the ratchet boundary is required (CHEN and PONTER [9]). The following concerns the case where we assume that the load state, for a prescribed λ , is within the reverse plasticity load region. First, the functional, $I(\dot{\varepsilon}_{ij}^c, \lambda)$, is minimised for mechanisms that satisfies $\Delta\varepsilon_{ij}^c = 0$, but taking into account Restriction 1, Eq. (2.5), of the minimum theorem and assuming that $\rho_{ij}^c(x, t)$ is finite. For problems that exhibit a reverse plasticity shakedown limit, this consists of increasing λ above this limit in increments. This is then followed by the evaluation of a load factor $\bar{\lambda}$ so that an appropriate constant load $\bar{\lambda}\bar{P}_i$ takes the load state to the ratchet limit. This is given by the shakedown limit with the elastic solution $\lambda\hat{\sigma}_{ij}$ now replaced by $\lambda\hat{\sigma}_{ij} + \rho_{ij}^c$, [8, 9].

Consider, again, the case where the elastic solution varies proportionally between two extreme values, $\hat{\sigma}_{ij}(t_1)$ and $\hat{\sigma}_{ij}(t_2)$, describing a straight-line path in stress space. Using this simplification, Eqs. (2.4) and (2.5) now give

$$(5.1) \quad \int_0^{t_1} \dot{\varepsilon}_{ij}^c dt = \Delta\varepsilon_{ij}^1 \quad \text{and} \quad \int_{t_1}^{t_2} \dot{\varepsilon}_{ij}^c dt = \Delta\varepsilon_{ij}^2 = -\Delta\varepsilon_{ij}^1$$

and

$$(5.2) \quad \int_0^{t_1} \dot{\rho}_{ij}^c dt = \Delta\rho_{ij}^1 \quad \text{and} \quad \int_{t_1}^{t_2} \dot{\rho}_{ij}^c dt = \Delta\rho_{ij}^2 = -\Delta\rho_{ij}^1,$$

$$\Delta\varepsilon_{ij}^1 + \Delta\varepsilon_{ij}^2 = 0, \quad \Delta\rho_{ij}^1 + \Delta\rho_{ij}^2 = 0.$$

Thus, by taking into consideration only $\Delta\varepsilon_{ij}^1$ and $\Delta\rho_{ij}^1$, the extremes of the reverse-plasticity mechanism can then be identified. In the following, it is important to note that the strain increment, $\Delta\varepsilon_{ij}^1$, is assumed to occur at a fixed point on the yield surface, in contrast to the exact solution, where the plastic strain rate may move around the yield surface. For proportional loading problems we find (CHEN and PONTER [9]) that this assumption has little effect upon the solution and considerably simplifies the solution method.

As before, the evaluation of $\Delta\varepsilon_{ij}^1$ and $\Delta\rho_{ij}^1$ requires the solution of a sequence of linear problems. For an initial estimate, $\Delta\varepsilon_{ij}^1 = \Delta\varepsilon_{ij}^{1I}$, a new estimate,

$\Delta\varepsilon_{ij}^1 = \Delta\varepsilon_{ij}^{1F}$, is then defined as the solution of a linear problem corresponding to a shear linear coefficient, $\bar{\mu}^I$ being given by the matching condition for the range of stress,

$$(5.3) \quad 2\sigma_y = \left(\frac{3}{2}\right) 2\bar{\mu}^I \bar{\varepsilon}(\Delta\varepsilon_{ij}^{1I}).$$

The new distribution of the strain increment, $\Delta\varepsilon_{ij}^{1F}$, is then characterized as the solution to the following problem:

$$(5.4) \quad \Delta\varepsilon_{ij}^{1TF'} = \frac{1}{2\mu} \Delta\rho_{ij}^{1F'} + \Delta\varepsilon_{ij}^{1F'}, \quad \Delta\varepsilon_{kk}^{1TF} = \frac{1}{3K} \Delta\rho_{kk}^{1F}$$

and

$$(5.5) \quad \Delta\varepsilon_{ij}^{1F'} = \left(\frac{1}{2\bar{\mu}^I}\right) (\Delta\hat{\sigma}_{ij}' + \Delta\rho_{ij}^{1F'}), \quad \Delta\varepsilon_{kk}^{1F} = 0,$$

where $\Delta\varepsilon_{ij}^{1TF}$ is kinematically admissible and $\Delta\rho_{ij}^{1F}$ satisfies equilibrium conditions for zero applied loads. Here

$$(5.6) \quad \Delta\hat{\sigma}_{ij} = \lambda(\hat{\sigma}_{ij}^\theta(t_1) - \hat{\sigma}_{ij}^\theta(t_2)),$$

Note that the shear modulus, μ , and the bulk modulus, K , correspond to (isotropic) elastic material behaviour. In an iterative process, the repeated application of this algorithm produces a sequence of solutions for $\Delta\varepsilon_{ij}^{1F}$, which converge to the absolute minimum of the functional I [8, 9].

The numerical procedure now used to identify the ratchet limit, is the same as the upper bound shakedown limit discussed in the previous section. The linear elastic solution is replaced by

$$(5.7) \quad \hat{\sigma}_{ij} = \lambda\hat{\sigma}_{ij} + \bar{\lambda}\hat{\sigma}_{ij}^{\bar{P}}(x_i) + \rho_{ij}^r(x_i, t),$$

where $\hat{\sigma}_{ij}^{\bar{P}}(x_i)$ denotes the linear elastic solution corresponding to the additional loads $\bar{P}_i$. The residual history $\rho_{ij}^r(x_i, t)$ is given by

$$\begin{aligned} \rho_{ij}^r(x_i, 0) &= 0, & \rho_{ij}^r(x_i, t_1) &= \Delta\rho_{ij}^1, \\ \rho_{ij}^r(x_i, t_2) &= \Delta\rho_{ij}^1 + \Delta\rho_{ij}^2 = 0 = \rho_{ij}^r(x_i, \Delta t). \end{aligned}$$

6. Examples of applications

In the following we describe three examples from differing applications; the shakedown of surfaces under rolling contact, the performance of a heat exchanger subjected to severe thermal loading and the performance of a particulate metal matrix composite subjected to variable temperature.

6.1. Example 1. The shakedown of a surface subjected to repeated rolling contact

A half-space, $z > 0$, is composed of an elastic perfectly plastic material. A contacting body rolls, repeatedly from $x = -\infty$ to $x = +\infty$. The contact area is elliptical with semi axes a and b as shown in Fig. 1. Within this area normal pressure $p(x, y)$ is assumed to be given by the solution of the Hertz contact problem. In addition a frictional force acts in the x direction given by

$$q(x, y) = fp(x, y),$$

where f is a coefficient of friction. Linear elastic solutions have been given by HAMILTON [12] and SACKFIELD and HILL [13]. Solutions for the shakedown solution to this problem were given, using a semi-analytic method, by PONTER HEARLE and JOHNSON [11], mainly for the case of circular contact, $a = b$.

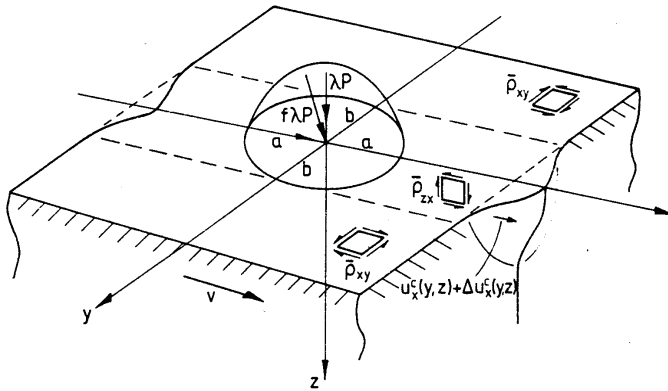


FIG. 1. A rolling contact problem.

Details of the finite element implementation may be found in CHEN and PONTER [14]. Any ratchet mechanism will be independent of the x direction. A finite element mesh is chosen in the (y, z) plane of three-dimensional brick elements. Displacement constraints are then applied to ensure that the displacement on the opposing $x = \text{constant}$ surfaces are identical. The cycle of loading then consists of positioning the contact region at a sequence of positions along the x -axis, thereby generating an elastic stress history for each element Gauss point.

Shakedown limits are shown as an interaction diagram in Fig. 2. Here $k = \sigma_y/\sqrt{3}$ denotes the pure shear yield stress. For a fixed value of the contact ellipse aspect ratio b/a , the shakedown limit is shown as a contour in a space where the axes are given by the normal load and the traction coefficient f . The extreme case of line contact, $b/a \rightarrow \infty$ may be evaluated directly from

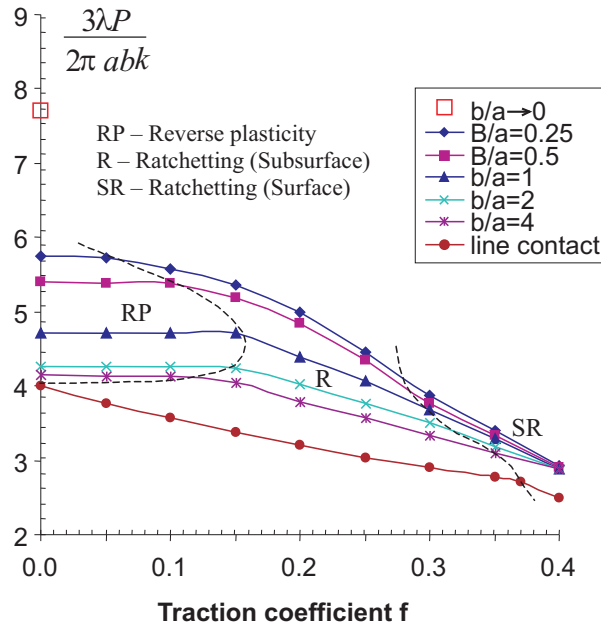


FIG. 2. Shakedown boundaries for the rolling contact problem for a range of ellipse aspect ratios b/a .

the two-dimensional line contact solution as the optimum mechanism consists of a slip surface parallel to the contact surface. The solution for $b/a = 1$ corresponds closely to the solution given by PONTER, HEARLE and JOHNSON [11]. By inspecting the mechanisms in each optimal solution at convergence, it is possible to classify the mechanisms into three broad categories:

1. RP – Reverse plasticity mechanism, where plastic strains occur at a single point (numerically at a single Gauss point), with zero plastic strains elsewhere.
2. R – A subsurface ratchet mechanism, similar in form to the mechanism chosen by PONTER, HEARLE and JOHNSON [11] although generally of a more complex nature than the simple form assumed by these authors.
3. SR – Surface ratchetting. For these cases the mechanism occurs at the surface and consist of the movement, in the direction of travel, of a thin surface layer within the contact area.

It can be seen that reverse plasticity (RP) occurs for low friction coefficient, less than $f \cong 0.15$, except for large and small values of b/a . Surface ratchetting occurs for high values of $f > 0.25$. For the intermediate range, where realistic f values occur, the mechanism is a ratchet mechanism beneath the surface.

The extreme case of $b/a = 0$ and $f = 0$ reduces to a two-dimensional limit load problem, and the corresponding value is included for comparison. The value is significantly higher than the computed case of $b/a = 0.25$.

This example demonstrates that the method is capable of providing reference solutions to complex problems. Such rolling contact solutions have application in the design of railway lines, bearing surfaces and road pavements.

6.2. Example 2. A heat exchanger subjected to severe thermal cycling

Figure 3 shows a 1/16-th section of a heat exchanger from a power plant. Such exchangers are subjected to particular severe thermal loading, resulting in the possibility of ratchetting and premature failure due to high cycle fatigue.

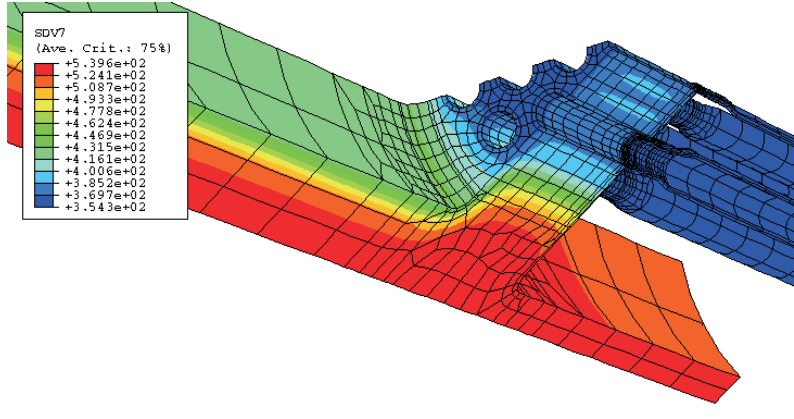


FIG. 3. Heat Exchanger. Temperature distribution at Boiler Trip °C.

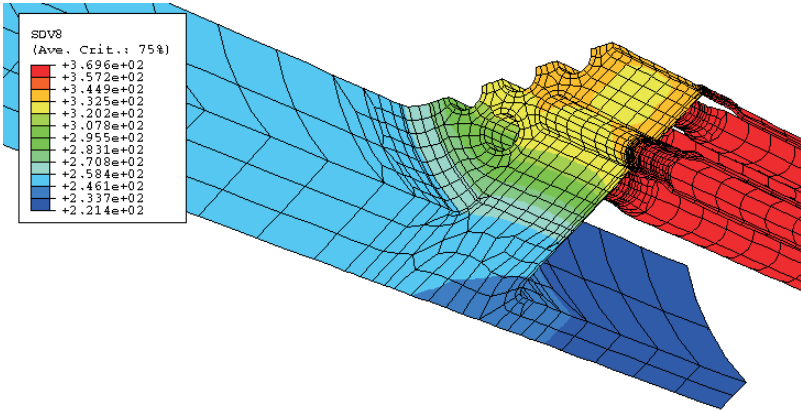
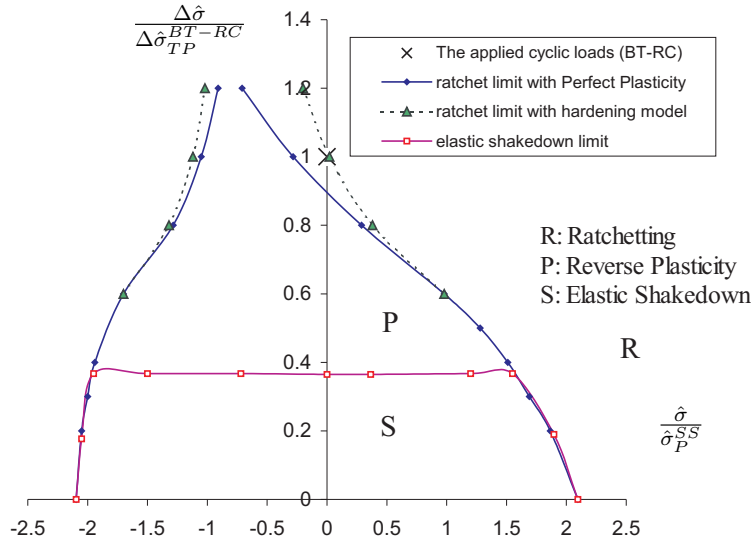


FIG. 4. Heat Exchanger. Heat Exchanger. Temperature at Boiler Reconnect, °C.

In Figs. 3 and 4 are shown two extreme temperature distributions that occur when the superheated steam supply is suddenly disconnected (Boiler Trip) and when the superheated steam supply is reconnected (Boiler Reconnect). At the same time there is a varying internal steam pressure. If this sequence is regarded as a cycle of loading, the resulting interaction diagram may be evaluated, using the methods of Secs. 4 and 5, as shown in Fig. 5. The linear elastic stress histories were evaluated and the maximum variation of effective elastic stress was denoted by $\Delta\hat{\sigma}_{TP}^{BT-RC}$. This linear solution was then scaled and the vertical axis of Fig. 5 $\Delta\hat{\sigma}/\Delta\hat{\sigma}_{TP}^{BT-RC}$ corresponds to differing scaling factors where $\Delta\hat{\sigma}/\Delta\hat{\sigma}_{TP}^{BT-RC} = 1$ corresponds to the actual history. The horizontal axis $\hat{\sigma}/\hat{\sigma}_P^{SS}$ corresponds to the maximum elastic effective stress for an internal pressure, where $\hat{\sigma}/\hat{\sigma}_P^{SS} = 1$ corresponds to the internal pressure experienced by the heat exchanger in normal operation. Variation of the yield stress with temperature was taken into account as this has a significant effect on the solutions. The diagram is subdivided into regions where shakedown (S), reverse plasticity (P) and ratchetting (R) occurs. It is possible to adapt the method so that cyclic hardening is taken into account and this effects the position of the ratchet boundary. Using the known steady state cyclic behaviour for the material (an austenitic stainless steel), the corresponding ratchet boundary is shown in Fig. 5 as a dashed line.



$\Delta\hat{\sigma}_{TP}^{BT-RC}$ maximum elastic stress range from Boiler Trip (BT) to Reconnect (RC).
 $\hat{\sigma}_P^{SS}$ maximum elastic internal pressure at the steady state normal operation.

FIG. 5. Interaction diagram for Heat Exchanger problem of Fig. 4, showing the ranges of loading for which differing modes of behaviour occur.

This method of representing the behaviour of the structure can be seen to have considerable advantages. The actually loading history, $\Delta\hat{\sigma}/\Delta\hat{\sigma}_{TP}^{BT-RC} = 1$, lies slightly outside the ratchet boundary within the ratchet region, assuming perfect plasticity. When cycle hardening is taken into account, the load point lies on the ratchet boundary. This characteristic of the problem corresponds very well with the known behaviour of the component. The load point lies well outside the shakedown region and this is typical of such problems where the amplitude of plastic strain and the location of the load point in relation to the ratchet boundary are of greatest interest. A full discussion of the solutions and comparisons with step-by-step solutions for complex constitutive equations are given by CHEN and PONTER [15, 16].

This example demonstrates that, for these complex industrial problems, the method is capable of providing solutions that are much more illuminating than conventional analysis.

6.3. Example 3. A particulate metal matrix composite subjected to constant stress and variable temperature

The final example concerns the behaviour of a metal matrix composite material. Such materials consist of a combination of a ductile matrix metal within which is incorporated, in a regular manner, a ceramic. The ceramic may be in the form of long continuous fibres or particles. Such materials have higher strength, greater stiffness and lower density than the monolithic matrix material and hence are potentially advantageous for aerospace applications. An aspect of such materials that is potentially difficult to understand in all its aspects is the effect of variable temperature. The constitutive components, ceramic and metal, have significantly differing coefficients of thermal expansion and this gives rise to micro thermal stresses when the (uniform) temperature of the material is changed.

We consider a regular particulate composite that consists of a regular three dimensional array of cubic elements. A particle of the ceramic is positioned at the centre of each cube. In Fig. 6 we show the finite element mesh for one eighth of a generic cube for two values of V_p , the volume fraction occupied by the ceramic particle.

The following problem was considered. A uniaxial macro-stress Σ is applied in a direction parallel to an edge of the generic cube and maintained constant. The temperature of the composite remains uniform but varies cyclically over a range θ_0 to $\theta_0 + \Delta\theta$. The generic cube is subjected to homogenisation boundary conditions so that the surface displacement in a single cube is consistent with that of adjacent, identical cubes.

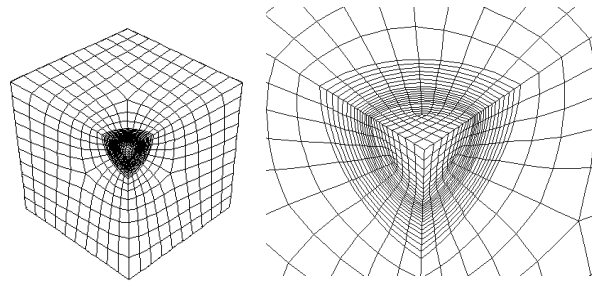
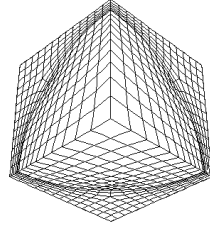
a) Mesch $V_p = 0.052\%$ b) Mesch $V_p = 38.17\%$

FIG. 6. Typical finite element meshes for $1/8^{\text{th}}$ of the generic cube of a regular particulate metal matrix composite. The elements are 20 noded quadratic.

Figure 7 shows a sequence of interaction diagrams for a range of V_p where the axis are expressed in non-dimensional variables, Σ/Σ_y and $\Delta\sigma^\theta/\Delta\sigma_{RP}^\theta$. Here Σ_y denotes the limit macro-stress Σ of the composite for $\Delta\theta = 0$; $\Delta\sigma^\theta$ denotes the

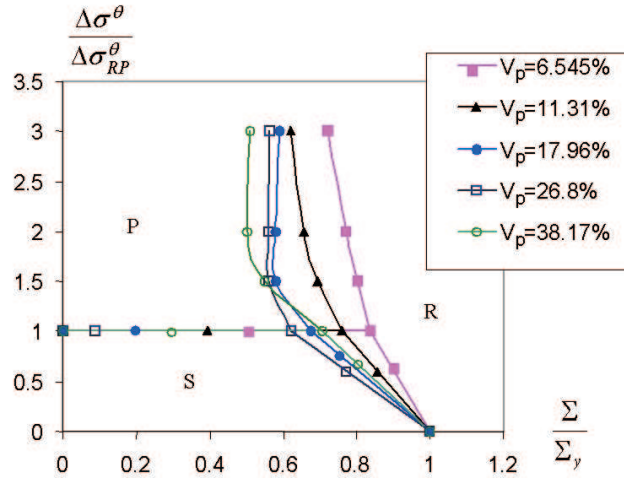


FIG. 7. Shakedown limits and ratchet limits for metal matrix composite consisting of an aluminium (Al) matrix with the particles of alumina (Al_2O_3) for a range of V_p , the proportion of the volume occupied by the alumina particles.

maximum effective elastic stress due to $\Delta\theta$ whereas $\Delta\sigma_{RP}^\theta$ denotes the value of $\Delta\sigma^\theta$ at the reverse plasticity shakedown boundary between the regions S and P. Figure 7 was evaluated using appropriate material properties for an aluminium matrix and alumina particles. Solutions for silica carbide particles produced a nearly identical diagram; this set of non-dimensional variable provides a diagram that will be reasonable accurate for any combination of elements.

The most noticeable feature of Fig. 7 is the observation that the reduction of the effective strength of the composite due to variable temperature has a saturation value, which varies with volume fraction V_p with a maximum reduction of about 50%.

7. Conclusions

The paper concentrates on the behaviour of an elastic/perfectly plastic body subjected to cyclic loading. The approach involves the characterisation of the cyclic state as a minimum principle. For an approximating class of kinematically admissible strain rate histories, an optimal minimum may then be found by a simple programming method, the Linear Matching Method. The objective of this study is to demonstrate that Direct Methods may be applied to a much wider range of circumstances than have hitherto been possible. The methodology has been applied to complex problems involving both creep and plasticity [15, 16]. Minimum theorems may be derived for cyclic creep and Direct Methods for cyclic creep problems [17, 18] have been solved, for the first time, by the methodology described in this paper.

Acknowledgement

The authors acknowledge the support of the Engineering and Physical Sciences research Council, British Energy Ltd and the University of Leicester during the course of the work reported in this paper. They would also like to express thanks to G Specchia of the Politecnico di Bari, Italy, for assistance in generating the data for Fig. 2 and Alex Savoury of the Department of Engineering, Leicester University, for the analytic solution of line contact.

References

1. W. T. KOITER, *General theorems of elastic-plastic solids*, [in:] J. N. SNEDDON, R. HILL [Eds.], *Progress in Solid Mechanics*, **1**, 167–221, 1960.
2. D.L. MARRIOTT, *Evaluation of deformation or load control of stresses under inelastic condition using elastic finite element stress analysis*. Proc. ASME Pressure Vessel and Piping Conference, Pittsburg, PA, PVP-**136**, 3–9, 1998.

3. R. SESHADRI, S. P. MANGALARAMANAN, *Lower bound limit loads using variational concepts: the m_α – method*, International Journal of Pressure Vessels and Piping, **71**, 93–106, 1997.
4. J. T. BOYLE, R. HAMILTON, J. SHI, D. MACKENZIE, *Simple method of calculating lower-bound limit loads for axisymmetric thin shells*, J. of Pressure Vessel Tech., **119**, 236–24, 1997.
5. H. CHEN and A. R. S. PONTER, *Application of the linear matching method to the integrity assessment of the high temperature response of structures*, [in:] Proceedings of American Society of Mechanical Engineering, PVP-441, 185–196, 2003.
6. R. A. AINSWORTH, P. J. BUDDEN, *Approximate inelastic analysis of defective components*, Nucl. Engrg. Design, **133**, 513–523, 1992.
7. R. A. AINSWORTH [Ed.], *R5: Assessment procedure for the high temperature response of structures. Issue 3*. British Energy Generation Ltd, Barnwood, Gloucester, UK, 2003,
8. A. R. S. PONTER, and H. F. CHEN, *A minimum theorem for cyclic loading in excess of shakedown, with applications to the evaluation of a ratchet limit*, European Journal of Mechanics A/Solids, **20**, 539–554, 2001.
9. H. F. CHEN and A. R. S. PONTER, *A method for the evaluation of a ratchet limit and the amplitude of plastic strain for bodies subjected to cyclic loading*, European Journal of Mechanics A/Solids, **20**, 555–572, 2001.
10. A. R. S. PONTER and M. ENGELHARDT, *Shakedown limits for a general yield condition: Implementation and examples for a von Mises yield condition*, European Journal of Mechanics, A/Solids, **19**, 423–446, 2000.
11. A. R. S. PONTER, A. D. HEARLE, K. L. JOHNSON, *Application of the kinematical shakedown theorem to rolling and sliding point contacts*, J. Mech. Phys. Solids, **33**, 339–362, 1985.
12. G.M. HAMILTON, *Explicit equations for the stresses beneath a sliding spherical contact*. Proc. Instn. Mech. Engrs., **197C**, 53–59, 1983.
13. A. SACKFIELD, and D. A. HILLS, *Some useful results in the classical Hertz contact problem*, Journal of Strain Analysis, **18**, 101–105, 1983.
14. H. F. CHEN and A. R. S. PONTER, *The linear matching method for shakedown and limit analyses applied to rolling and sliding point contact problems*, International Journal of Road Materials and Pavement Design, 2005 (to appear).
15. H. F. CHEN and A. R. S. PONTER, *Integrity assessment of a 3D tubeplate using the linear matching method, Part 1: Shakedown, reverse plasticity and ratchetting*, Int. J. Pressure Vessels and Piping, **82**, 85–94, 2005.
16. H. F. CHEN and A. R. S. PONTER, *Integrity assessment of a 3D tubeplate using the linear matching method, Part 2: Creep relaxation and reverse plasticity*, Int. J. Pressure Vessels and Piping, **82**, 95–104, 2005.
17. A. R. S. PONTER and M. BOULBIBANE, *Minimum theorems and the linear matching method for bodies in a cyclic state of creep*, European Journal of Mechanics, A/Solids, **21**, 899–914, 2003
18. M. BOULBIBANE and A. R. S. PONTER, *A method for the evaluation of design limits for structural materials in a cyclic state of creep*, European Journal of Mechanics, A/Solids, **21**, 915–925, 2003.

19. D. A. GOKHFELD and O. F. CHERNIAVSKY, *Limit analysis of structures at thermal cycling*. Sijthoff and Noordhoff, Alpen aan den Rijn, The Netherlands, 1980.
20. C. POLIZZOTTO, *Variational methods for the steady state response of elastic-plastic solids subjected to cyclic loads*, Int. J. Solids Structures, **40**, 2673–2697, 2003.

Received October 18, 2004; revised version February 1, 2005.
