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Brief Note

A note on the flows of inhomogeneous fluids with shear-dependent viscosities

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Inhomogeneous fluids have not been studied with the intensity that they deserve. In fact, many studies that are supposedly concerned with the response of inhomogeneous fluids are not directed at inhomogeneous fluids, and this stems from not recognizing the fact that the properties of a fluid varying in its current configuration does not mean that the fluid is inhomogeneous. Here, we show that mild variations in the properties of the fluid which might warrant it being approximated as a homogeneous fluid with average properties could lead to significant errors in the computation of both global and local quantities, associated with the flow.

1. Introduction

Whether fluid or solid, most if not all bodies are inhomogeneous. When the inhomogeneity is "sufficiently mild", we ignore the inhomogeneity and model the body as a homogeneous body. In this paper, we are interested in investigating the errors inherent in such an approximation within the context of nonlinear fluids. We find that approximating the properties of an inhomogeneous fluid that varies mildly about a mean value can lead to differences in global responses that can differ much more than the variation in the value of the property would suggest. Thus, when the viscosity of a fluid varies by, say 2% about the mean, the flow rate may vary by 20 to 30%, i.e., larger by an order of magnitude. Local quantities such as the shear stresses (or velocity gradients) can vary more significantly (several orders of magnitude). We find that the maximum shear rate for the flow of an inhomogeneous fluid whose viscosity varies by 2% about a mean can exhibit maximum shear rates that vary significantly (say 4500%) from the maximum shear rate in the case of the corresponding homogeneous fluid. These results can have profound implications on the flow of inhomogeneous geological and biological fluids which we might choose to approximate as homogeneous fluids for the sake of computational and analytical convenience. We illustrate the main ideas by means of a very simple boundary value problem, with very simple forms for the inhomogeneity, but this should not detract from the value of the work. To illustrate the ideas within the context of a flow in a more complicated geometry will tend to obscure the basic issues, namely that slight inhomogeneities can lead to very large differences in local measures for the flow.

Let $\mathcal{B}$ denote an abstract body and let $P \in \mathcal{B}$ denote a typical material point. A one-to-one invertible mapping κ that assigns to each point $P \in \mathcal{B}$ a point $\mathbf{X} \in \kappa(\mathcal{B})$ in a three-dimensional Euclidean space is called a placer and $\kappa(\mathcal{B})$ is called a configuration of the body. Let $\kappa_t(\mathcal{B})$ denote the configuration of the body at say some time t. If t is the current time then $\kappa_t(\mathcal{B})$ would be its current configuration. Let $\mathbf{x} \in \kappa_t(\mathcal{B})$ denote a typical point belonging to the current configuration. The fact that the properties of a body vary from one point to another in $\kappa_t(\mathcal{B})$ does not mean that the body is inhomogeneous. **This point cannot be overemphasized**. Unfortunately, in much of the literature in fluid mechanics devoted to inhomogeneous fluids, sufficient care is not exercised in explaining what is meant by an inhomogeneous fluid as the equations are invariably expressed from an Eulerian point of view (the fact that the density of a fluid varies with location due to the action of gravity does not mean that the fluid is inhomogeneous). For example, it is possible that the properties of a homogeneous fluid could vary from one point to another in the current configuration. To recognise this consider a homogeneous[1)] shear thinning fluid, homogeneous in some reference configuration, say a static initial configuration, that starts undergoing a complex flow, the shear rate varying from one point in the flow domain to another. In this case the viscosity of the fluid would vary from one point to another in the flow domain but this by no means implies that the fluid is inhomogeneous. The point is, in general, that an Eulerian description cannot be used to determine whether a body is homogeneous, it is a notion that is inherent to the abstract body $\mathcal{B}$ and not its present configuration, or for that matter any specific configuration. This problem cannot be overcome by opting for a Lagrangian description as what one chooses as the reference configuration is yet arbitrary. In fact, the Eulerian perspective is nothing but a special Lagrangian perspective where the current configuration is considered as the reference configuration. Usually, one refers to using the initial configuration as the reference configuration as the Lagrangian perspective. In short, homogeneity is a notion associated with the abstract body $\mathcal{B}$ and does not refer to response from any specific configuration. Also, one could very well ask if the current definition of homogeneity that implies the existence of "some configuration" in which the properties of the fluid are constant, has any value whatsoever as only the configurations that the fluid actually takes matter. We shall not get into a discussion of whether the definition of an inhomogeneous body, as it is commonly used now,

[1)] We give a rigorous definition of what is meant by a homogeneous body, below.

is useful and if a better definition can be put in place. We shall work with the current definition of an inhomogeneous body.

Before we define what we mean by an inhomogeneous body, we first define what we mean by a materially uniform body. Two material points $P_1, P_2 \in \mathcal{B}$ are said to be materially uniform (see Truesdell and Noll [10]), when attention is restricted to purely mechanical issues, if there exist two placers κ_1 and κ_2 of the abstract body $\mathcal{B}$ into a three-dimensional Euclidean space such that there exist neighbourhoods $N_{\mathbf{X_1}}$ of $\mathbf{X_1} := \kappa_1(P_1)$ and $N_{\mathbf{X_2}}$ of $\mathbf{X_2} := \kappa_2(P_2)$ that are indistinguishable with regard to their mechanical response. If all the particles belonging to the body are pairwise materially uniform, the body is said to be materially uniform. A body is said to be homogeneous if all the material points belonging to the body are materially uniform with respect to a single placer κ. A body that is not homogeneous is said to be inhomogeneous. The above definition has its own shortcomings as it requires one to know what we mean by the abstract body $\mathcal{B}$. While we know the configuration of this body $\mathcal{B}$ from the place that it occupies in a three-dimensional Euclidean space at some instant of time, the body $\mathcal{B}$ is in of itself a primitive set. Any configuration of the body can serve as a reference and we are yet left with the onerous task of having to show that there exists a configuration of the body in which its properties do not vary for us to define the body as being homogeneous[2)]. It is only if one can show that there exists no configuration $\tilde{\kappa}(\mathcal{B})$ in which the properties do not vary, can one conclude that the body is inhomogeneous. From a practical standpoint, it is well nigh impossible to prove that there is no other configuration in which the properties do not vary, for even if we were to carry out an infinity of experiments, we cannot be sure that there is not one other configuration where the properties are a constant.

It is important to understand the difference between a body being homogeneous and a deformation being homogeneous. A deformation is said to be homogeneous if in a Cartesian coordinate system, the deformation gradient has constant entries.

There have been many studies concerning flows in which material particles remain on specific planes ("horizontal strata"). While properties like density and viscosity are allowed to vary from one strata to another, they do not vary on the

[2)]When working with biological materials that are growing, it is necessary to discuss the notion of inhomogeneity of bodies and here, as the body is constantly adapting and growing, the body is not a fixed set of particles and one only has an Eulerian description of the body as other particles that were once present, may not be in place at the current time. In this case the notion of inhomogeneity is far more complicated and one can use the Eulerian description, fix a set of particles that correspond to the configuration in question to get at the abstract body of interest that corresponds to the real body of interest that exists at time t. We shall not get into a discussion of such a definition here.

strata. As the particle remains on the plane, using the coordinate perpendicular to the plane along which the properties vary, one can use an Eulerian coordinate to carry out Lagrangian tracking, as the same coordinate normal to the strata serves as both the Eulerian and Lagrangian coordinate. In general such a device cannot be adopted if one is to study general three-dimensional flows of an inhomogeneous fluid. One has to start with some configuration (κ_R) in which the inhomogeneity is defined, and then determine which material particle $\mathbf{X}$ belonging to the original reference configuration presently occupies a specific point $\mathbf{x}$ in the current configuration (κ_t). As particles are always required to remain on a horizontal plane, we can conclude, if we require configurations of the body to be those actually taken by the body but not those in which the body could be possibly placed, then we could conclude there is no configuration in which the body can have uniform properties. More importantly, just because in the current configuration due to the action of gravity, the density of a fluid varies, it does not make the fluid an inhomogeneous fluid. The point is, in the absence of gravity the density might not vary and the fluid could be homogeneous. Thus, studies of stratified fluids, while perfectly reasonable, are not necessarily studies of inhomogeneous fluids. It is possible that one could indeed have stratified flows of an inhomogeneous fluid but this has to be decided on the basis of the body under consideration and not on the basis of the property of the fluid, such as density, varying due to body force fields such as gravity or for that matter any type of force.

There has been a considerable amount of work on the flow of fluids (see Yih [11, 12]) that are primarily concerned with stratified Euler fluids, purely due to the variation in the density (in fact there is no treatment of inhomogeneous Navier–Stokes fluids in [11] or [12]. The only problem, where frictional forces enter into consideration, is in the study of flow through porous media where the friction at the pores is included. However, the friction in the fluid itself is neglected).

Rayleigh [6] studied the character of the equilibrium of an incompressible Euler fluid arranged in horizontal strata, its density depending on the co-ordinate normal to the horizontal strata, by subjecting it to perturbations. It might be possible to place the fluid in a configuration, namely one which corresponds to the effect of gravity being negated (i.e., deforming the current configuration appropriately) wherein the density is a constant. Moreover, the same fluid in a zero gravity environment would have constant density. This is not to say one cannot study the flows of fluids under the influence of gravity that causes the density to vary. Studies of stratified fluids concern such flows and are not necessarily flows of inhomogeneous fluids. Of course it is important to point out that Rayleigh does not call the fluids inhomogeneous. He is concerned with “stratified flows” which he defines precisely. The perturbations were assumed by Rayleigh to be small and dependent on x, y and z. He considered the stability of two homogeneous fluids of constant density separated by a common boundary. Harrison [3] studied two linearly viscous fluids in the same geometry, in order to determine the effect of viscosity on the character of the stability. This study seems to be one of the earliest studies on the flow of inhomogeneous Navier–Stokes fluids. Other early studies involving inhomogeneous Navier–Stokes fluids are those by Lewis [5], Taylor [9], Hide [4] and Chandrasekhar [1, 2]. Like Harrison [3], Chandrasekhar [2], generalized Rayleigh's treatment of the case of two uniform fluids separated by a horizontal boundary to the case of an incompressible Navier–Stokes fluid by allowing the viscosity to depend on the vertical coordinate. He however carried the instability analysis a lot further. While there have been a few other studies concerning inhomogeneous Navier–Stokes fluids, there is the need for a careful and systematic study of such fluids.

Here, we shall consider fluids whose density is constant but whose viscosity varies from point to point in its reference configuration, i.e., we are interested in inhomogeneous incompressible fluids. We shall also allow the fluid to be such that its generalized viscosity can allow it to shear thin or shear thicken, that is the viscosity will be assumed to depend on the symmetric part of the velocity gradient. The form that we have chosen for the generalized viscosity allows for a variety of characteristics for the viscosity depending on the values that can be assigned to the parameters that appear in its definition.

We consider models wherein the generalized viscosity displays characteristics associated with real fluids. To illustrate our thesis, we study the flow between two parallel plates due to a pressure gradient and that between two concentric circular cylinders, namely plane Poiseuille flow and cylindrical Couette flow.

It is important to recognize that qualitative results that are established in this note are not a consequence of the special choice of the inhomogeneity. They have been chosen to make our ideas transparent within the context of a simple model. Finally, we observe that in the case of isotropic inhomogeneous nonlinear elastic solids, significant differences in the value of the stresses can exist between those in the actual inhomogeneous body and the homogenized approximation arrived at by obtaining a model that has an equivalent stored energy (see Saravanan and Rajagopal [7, 8]).

2. The model

We shall consider an inhomogeneous fluid whose Cauchy stress $\mathbf{T}$ is related to the fluid motion in the following manner:

$$\mathbf{T} = -p\mathbf{1} + \mu(\mathbf{X})\left(1 + \beta\ \left(\frac{1}{2}\mathrm{tr}\mathbf{A_1}^2\right)\right)^m \mathbf{A_1}\ , \tag{2.1}$$

where $-p\mathbf{1}$ is the indeterminate spherical stress due to the constraint of incompressibility, and $\mathbf{A_1}$ denotes the symmetric part of the velocity gradient. We note that when $m = 0$, the model reduces to an incompressible inhomogeneous Navier–Stokes fluid.

We shall be interested in plane Poiseuille flow for which the motion takes the form:

$$x = X + u(Y)t , \qquad y = Y , \qquad z = Z , \tag{2.2}$$

and we shall consider a viscosity which is inhomogeneous in the following manner:

$$\mu(\mathbf{X}) = \hat{\mu}(Y) = \hat{\mu}(y) . \tag{2.3}$$

We also consider cylindrical Couette flow wherein the motion is given by:

$$r = R , \qquad \theta = \Theta + \omega(R)t , \qquad z = Z , \tag{2.4}$$

and we assume that

$$\mu(\mathbf{X}) = \hat{\mu}(R) = \hat{\mu}(r) . \tag{2.5}$$

In the case of flow between two plates that are a distance 'h' apart we assume that

$$\hat{\mu}(Y) = \hat{\mu}(y) = \mu_0\left(1 + \epsilon\left(\frac{6y^2}{h^2} - \frac{1}{2}\right)\right) , \tag{2.6}$$

and also obtain results for another (similar) form:

$$\mu = \mu_0\left(1 + \epsilon\left(\frac{1}{2} - \frac{6y^2}{h^2}\right)\right) . \tag{2.7}$$

The choice of the inhomogeneity ensures that the mean value of the viscosity is μ_0, and if $\epsilon = 0.1$, it follows that the viscosity is never more than 10% from its mean value.

In the case of the (Couette) flow in the annulus between two very long cylinders, we assume that:

$$\hat{\mu}(R) = \hat{\mu}(r) = \mu_0\left(1 + \epsilon\left(r^2 - \frac{3.31}{3}\right)\right) , \tag{2.8}$$

and also obtain results for another (similar) form:

$$\mu = \mu_0\left[1 + \epsilon\left(\frac{3.31}{3} - r^2\right)\right] , \tag{2.9}$$

so that once again the mean viscosity is μ_0 and the viscosity never varies more than 10% from the mean.

3. Poiseuille flow between flat plates

On neglecting the body force, the appropriate governing equation is

$$\left[\mu(y)\left(1+\beta(u')^2\right)^m u'\right]' = C \ , \tag{3.1}$$

where C is the pressure gradient along the x-direction that is a constant, and the prime denotes the derivative with respect to the argument.

If one assumes that the fluid adheres to the boundaries, then we need to enforce:

$$u(h/2) \ = \ u(-h/2) \ = \ 0 \ . \tag{3.2}$$

Let us define the volumetric flow rate across unit cross-sectional area through:

$$Q = \int_{-h/2}^{h/2} u(y)dy \ . \tag{3.3}$$

Let Q_H denote the mass flow rate corresponding to a homogeneous power law fluid whose zero shear rate viscosity has a constant value μ_0. Let us define the difference Q_e through:

$$Q_e \ = \ \frac{Q - Q_H}{Q_H} \ , \tag{3.4}$$

where Q_H corresponds to the flow rate of the homogeneous power-law fluid.

The non-dimensional parameters associated with the governing equation are as follows:

$$\hat{u} = \frac{u}{V}, \quad \hat{y} = \frac{y}{h}, \quad \hat{\beta} = \frac{\beta V^2}{h^2} \quad \text{and} \quad \hat{C} = \frac{Ch^2}{\mu_0 V}.$$

The governing equation (3.1) subject to the condition in Eq. (3.2) can be solved easily, and it can be shown that the error in the flow rate by assuming a homogeneous fluid with a mean viscosity can be as high as 43%. While this error in itself is quite large, it does mask a much deeper problem, namely errors in the wall shear rate that can be of the order of 4000%! This can be verified from Figs. 1–2, wherein the error in the shear rate between that for the inhomogeneous fluid and that for the homogenized fluid are plotted. Tables 1 and 2 document the error in the flow rate and the velocity gradient (at the wall) between the inhomogeneous fluid whose inhomogeneity varies in the manner defined through Eqs. (2.7) and (2.6) respectively, and the corresponding homogeneous fluid.

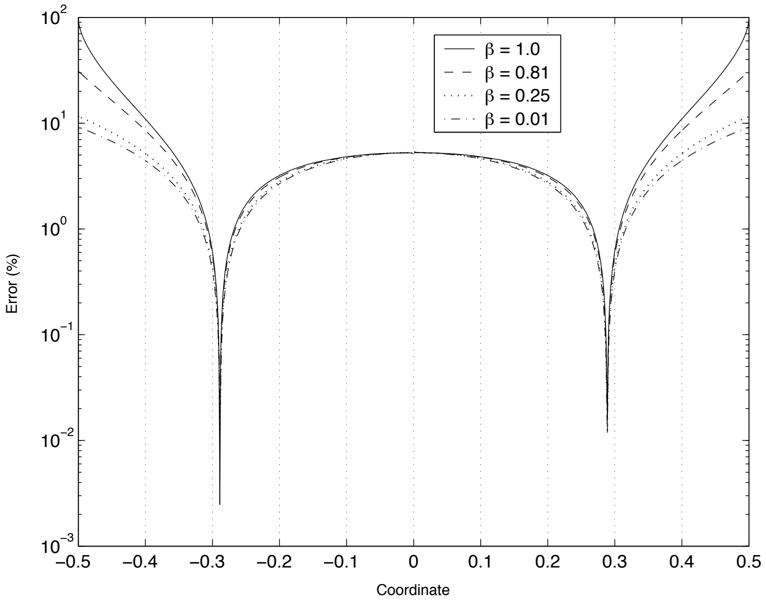

Fig. 1. Error (in the velocity gradient) for $m = -0.5$, $\hat{C} = -2.0$, $\epsilon = 0.1$. Inhomogeneity given in Eq. (2.6). Maximum error is 97.77%.

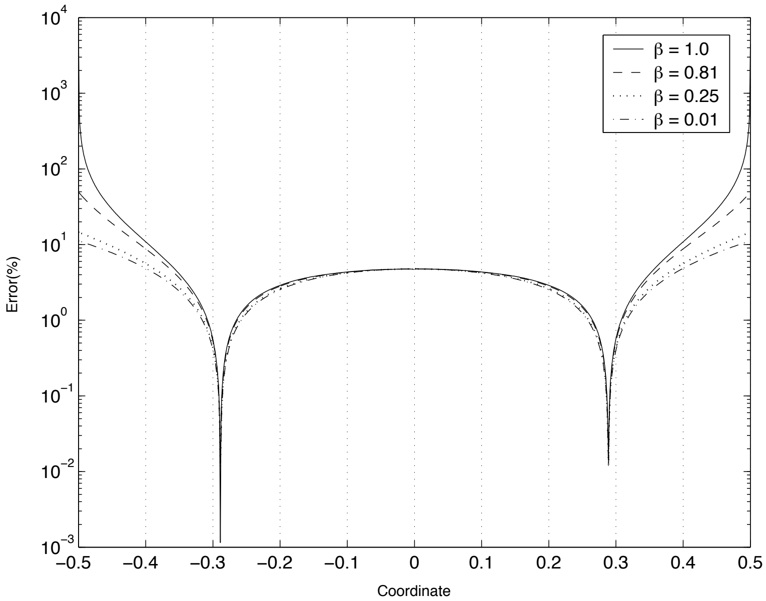

Fig. 2. Error (in the velocity gradient) for $m = -0.5$, $\hat{C} = -1.8$, $\epsilon = 0.1$. Inhomogeneity given in Eq. (2.7). Maximum error is 4517.66%.

Table 1. Error, in the flow rate and velocity gradient (at the wall), between the inhomogeneous fluid defined through Eq. (2.7) and the corresponding homogeneous fluid.

m	$\hat{C}$	Variable	$\hat{\beta}$			
			1.0	0.81	0.25	0.01
−0.5	−1.8	Error in the velocity gradient (at wall)	4517.66%	49.48%	14.58%	11.22%
−0.5	−1.8	Error in the mass flow	49.34%	14.17%	5.54%	4.38%
−0.4	−14.9	Error in the velocity gradient (at wall)	69.36%	69.34%	69.43%	20.12%
−0.4	−14.9	Error in the mass flow	42.83%	42.81%	42.85%	7.45%

Table 2. Error, in the flow rate and velocity gradient (at the wall), between the inhomogeneous fluid defined through Eq. (2.6) and the corresponding homogeneous fluid.

m	$\hat{C}$	Variable	$\hat{\beta}$			
			1.0	0.81	0.25	0.01
−0.5	−2.0	Error in the velocity gradient (at wall)	97.77%	31.07%	11.61%	9.17%
−0.5	−2.0	Error in the mass flow	34.60%	11.98%	4.76%	3.74%
−0.4	−15.0	Error in the velocity gradient (at wall)	37.91%	37.90%	37.89%	13.90%
−0.4	−15.0	Error in the mass flow	26.86%	26.86%	26.84%	5.75%

Given a specific model of the form (2.1), it is not possible to obtain solutions of the form (2.2), for all values of the pressure gradient C. Consider an integral of (3.1). We immediately recognize that, when say $m = -1/2$,

$$u' = \frac{\dfrac{Cy}{\mu(y)}}{\left(1 - \beta\left(\dfrac{Cy}{\mu(y)}\right)^2\right)^{1/2}} \,, \tag{3.5}$$

and in order for u' to be real, we need the denominator to be real which would not be possible for certain viscosities.

4. Couette flow in the annular region between two coaxial cylinders

The governing equations for the flow of the inhomogeneous fluid reduce to

$$\frac{d}{d\bar{r}}\left[\hat{\mu}(r)\left(1 + \beta r^2\left(\frac{d\omega}{dr}\right)^2\right)^m r^3\frac{d\omega}{dr}\right] = 0 \,, \tag{4.1}$$

and the adherence of the fluid to the two cylinders implies that:

$$\omega(R_i) = 0 \,, \qquad \omega(R_o) = \Omega \,, \tag{4.2}$$

where R_i and R_o are the radius of the inner and outer cylinders, respectively.

Let us suppose that

$$\frac{R_o - R_i}{R_i} = 0.1 \; . \tag{4.3}$$

For the sake of notational simplicity, let us define a constant $\bar{\beta}$ through $\bar{\beta} = \beta\Omega^2$.

Figure 3 records the variation of the error (with $\bar{\beta}$) in the velocity gradient (non-dimensional), between the inhomogeneous and homogeneous cases,

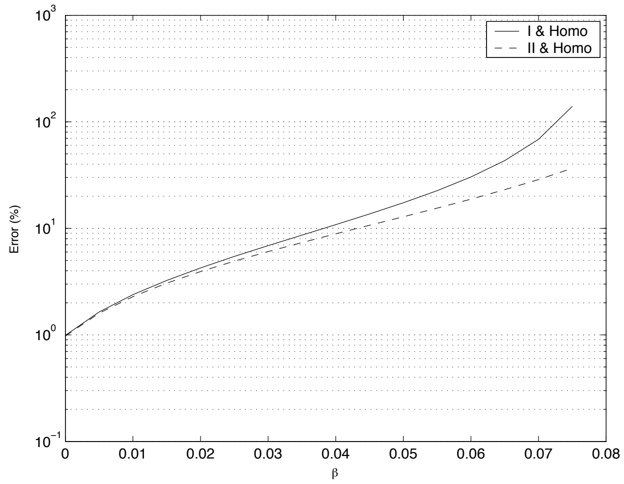

FIG. 3. Variation of error (in the velocity gradient at the inner cylinder) with $\bar{\beta}$ for $m = -0.5$, $\epsilon = 0.1$. Inhomogeneities defined through Eq. (2.8) – (I) and (2.9) – (II). Maximum errors are 139.67% and 36.58%, respectively.

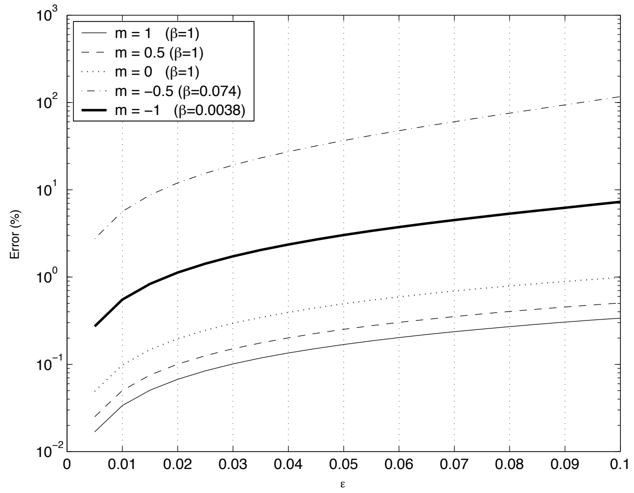

FIG. 4. Variation of error (in the velocity gradient at the inner cylinder) with ϵ for various m, $\bar{\beta}$. Inhomogeneities defined through Eq. (2.8) – (I).

recorded at the inner cylinder for a particular value of m. We see that the error increases with $\bar{\beta}$ and can be quite significant, i.e., greater than 100%. Figures 4 and 5 record the variation of this error with ϵ, and we see that a 10% variation in the viscosity can lead to an error over 100% in the shear rate.

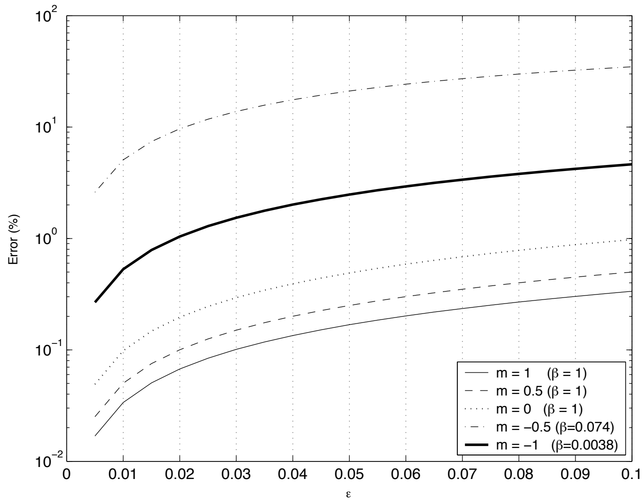

FIG. 5. Variation of error (in the velocity gradient at the inner cylinder) with ϵ for various m, $\bar{\beta}$. Inhomogeneities defined through Eq. (2.9) – (II).

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References

1. S. CHANDRASEKHAR, *The character of the equilibrium of an incompressible fluid sphere of variable density and viscosity subject to radial acceleration*, Quarterly Journal of Mechanics and Applied Mathematics, **8**, 1–21, 1955.

2. S. CHANDRASEKHAR, *The character of the equilibrium of an incompressible heavy viscous fluid of variable density*, Proceedings of the Cambridge Philosophical Society, **51**, 162–178, 1955.

3. W. J. HARRISON, *The influence of viscosity on the oscillations of superposed liquids*, Proceedings of the London Mathematical Society, **6**, 396–405, 1908.

4. R. Hide, *The character of the equilibrium of an incompressible heavy viscous fluid of variable density: An approximate theory*, Proceedings of the Cambridge Philosophical Society, **51**, 179–201, 1955.

5. D. J. Lewis, *The instability of liquid surfaces when accelarated in a direction perpendicular to their planes II*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, **202**, 1068, 81–96, 1950.

6. Lord Rayleigh, *Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density*, Proceedings of the London Mathematical Society, **14**, 170–177, 1883.

7. U. Saravanan, and K.R. Rajagopal, *A comparison of the response of isotropic inhomogeneous elastic cylindrical and spherical shells and their homogenized counterparts*, Journal of Elasticity, **71**, 1–3, 205–233, 2003.

8. U. Saravanan, and K.R. Rajagopal, *On the role of inhomogeneities in the deformation of elastic bodies*, Mathematics and Mechanics of Solids, **8**, 4, 349–376, 2003.

9. G. I. Taylor, *The instability of liquid surfaces when accelarated in a direction perpendicular to their planes I*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, **201**, 1065, 192–196, 1950.

10. C. Truesdell and W. Noll, S. Flugge [Ed.],, *The nonlinear field theories of mechanics*, Springer Verlag, Heidelberg 1965.

11. C.S. Yih, *Dynamics of non-homogeneous fluids*, Macmillan, New York 1965.

12. C.S. Yih, *Stratified flows*, Academic Press 2 edition, New York 1980.

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