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Influence of coupling through porosity changes on the propagation of acoustic waves in linear poroelastic materials

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In Memory of Professor Henryk Zorski

In earlier works it has been shown that linearization of thermodynamical models of poroelastic materials yields a contribution to stresses in the form $\beta\Delta_n$, $\Delta_n := n - n_E$, where n is the current porosity, n_E denotes its value in the thermodynamical equilibrium and β is a material parameter. It has been also claimed on the basis of rough estimates that this parameter gives only negligible contributions to the properties of acoustic waves. The purpose of this work is twofold. We investigate the influence of β on the propagation of acoustic waves in more details. We use the full linear model of saturated poroelastic materials in which an additional coupling to porosity changes appears. This results from the presence of a term with the porosity gradient in the momentum sources. Such a model becomes identical with Biot's model without any added mass contribution in the limit if infinite relaxation time of porosity $\tau \to \infty$ and for the parameters $\beta = 0$, and $N = 0$, where the latter is describing the above mentioned influence of porosity gradient on the momentum sources. The analysis of influence of the parameter N, i.e. the influence of the porosity gradient on properties of acoustic waves, is the second purpose of this work. We also indicate a correction of the permeability contribution to the momentum source. The permeability coefficient π is assumed in Biot's model to be dependent on the frequency. This is inconsistent with other temporal contributions to field equations. In order to obtain such a dependence in the Fourier space, one has to assume a viscous effect to enter the momentum source and we do so in the general equations. However, we do not use this general relation in the analysis of monochromatic waves.

1. Introduction

In numerous previous papers (e.g. [9, 10, 13]), it has been shown that the thermodynamical construction of nonlinear models of poroelastic materials yields a dependence of partial Cauchy stresses on the deviation of porosity from its value in the thermodynamical equilibrium. This is connected with the fact that such nonlinear models contain the porosity as an independent field described by its own balance equation.

It has been argued (e.g. [10]) that the linearization of such models yields the contribution to stresses in the form $\beta\Delta_n$, $\Delta_n := n - n_E$, where n is the current porosity, n_E denotes its value in the thermodynamical equilibrium and β is a material parameter. It has been claimed on a basis of rough estimates that this parameter gives only negligible contributions to the properties of acoustic waves[1)].

The purpose of this work is twofold. We investigate the influence of β on the propagation of acoustic waves in more details. However, we use the full linear model of saturated poroelastic materials in which an additional coupling to porosity changes appears. This results from the presence of a term with the porosity gradient in the momentum sources. As it has been shown in the work [14], such a contribution follows from the thermodynamical construction of a nonlinear model which reduces to Biot's model when linearized. This happens for the limit of infinite relaxation time of porosity $\tau \to \infty$ and for the parameters $\beta = 0$, and $N = 0$, where the latter is describing the above mentioned influence of the porosity gradient on momentum sources. The analysis of influence of the parameter N, i.e. the influence of the porosity gradient on the properties of acoustic waves is the second purpose of this work.

There is an additional difference of the linear models analyzed in this work and that constructed by M. A. Biot. It is a term with relative accelerations which is included in the momentum balance equations of Biot's model and neglected in the model investigated in this work. The reason is that we consider contributions of relative accelerations to be not essential for the wave analysis. A justification of this statement can be found in the recent paper [16].

We also indicate a correction of the permeability contribution to the momentum source. The permeability coefficient π is assumed in Biot's model to be dependent on the frequency. As indicated in the work [17], this is inconsistent with other temporal contributions to field equations. In order to obtain such a dependence in the Fourier space, one has to assume a viscous effect to enter the momentum source and we do so in the general equations. However, we do not use this general relation in the analysis of monochromatic waves. It is known that the above mentioned frequency dependence is not essential in the low frequency range, i.e. in applications of the analysis to soil mechanics. It has an influence in the high frequency range and it may be interpreted as an intervention of tortuosity. This influence, as indicated in [2], is quantitative but not qualitative and it seems to be not related to an influence of coupling through nonequilibrium contributions.

[1)] See: [12, 13], where the values $\beta = 0.313$ and 0.72 GPa are chosen, respectively. These are small in comparison with values of compressibility coefficients ($\sim$ 50 GPa) with which they enter additively the propagation conditions for acoustic waves.

2. Governing equations

In the case of isothermal processes, the most general linear model of poroelastic materials with the balance equation of porosity is based on the following balance equations:

- partial balance equations of momentum

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div} \mathbf{T}^S + \hat{\mathbf{p}}, \qquad \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = -\operatorname{grad} p^F - \hat{\mathbf{p}}, \tag{2.1}$$

- balance equation of porosity

$$\frac{\partial \Delta_n}{\partial t} + \mathit{\Phi}_0 \operatorname{div} \left(\mathbf{v}^F - \mathbf{v}^S\right) = \hat{n}, \tag{2.2}$$

- conservation of mass of the fluid component and the integrability condition for the deformation of the skeleton

$$\frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \qquad \frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \tag{2.3}$$

where $\mathbf{v}^S, \mathbf{v}^F$ denote the velocity fields of the skeleton and of the fluid, respectively, ρ_0^S, ρ_0^F are their constant initial mass densities, $\mathbf{T}^S$ is the partial Cauchy stress tensor in the skeleton, p^F is the partial pressure in the fluid, $\hat{\mathbf{p}}$ is the momentum source, n denotes the current porosity and n_E its equilibrium value, $\mathit{\Phi}_0$ is a material parameter, $\hat{n}$ denotes the source of porosity. ε denotes volume changes of the fluid and in the linear model it is related to changes of the partial mass density of the fluid ρ^F by the following relation:

$$\varepsilon = \frac{\rho_0^F - \rho^F}{\rho_0^F}. \tag{2.4}$$

The symmetric tensor $\mathbf{e}^S$ is the so-called Almansi–Hamel deformation tensor of the skeleton. Its trace describes changes of the partial mass density of the skeleton ρ^S

$$e = \frac{\rho_0^S - \rho^S}{\rho_0^S}, \qquad e := \operatorname{tr} \mathbf{e}^S. \tag{2.5}$$

The above equations become field equations for the following governing fields of the isotropic model

$$\left\{\mathbf{v}^S, \mathbf{v}^F, \varepsilon, \mathbf{e}^S, n\right\}, \tag{2.6}$$

if we add the following constitutive relations:

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S e \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1} + \beta \Delta_n \mathbf{1}, \\ p^F &= p_0^F - \rho_0^F \kappa \varepsilon - Qe + \beta \Delta_n, \end{aligned} \tag{2.7}$$

$$\hat{\mathbf{p}} = \int\limits_{-\infty}^{t} \pi\left(t-s\right)\cdot\left(\mathbf{v}^F - \mathbf{v}^S\right) ds - N \text{ grad } \left(n - n_0\right), \tag{2.8}$$

$$n_E = n_0\left(1 + \delta e\right), \qquad \hat{n} = -\frac{\Delta_n}{\tau}, \tag{2.9}$$

where

$$\left\{\lambda^S, \mu^S, \kappa, Q, \beta, N, \Phi_0, \tau, \delta\right\}, \tag{2.10}$$

are material constants and π is the time-dependent bulk permeability of the porous body. The classical Darcy law contains a material parameter called the hydraulic conductivity. This parameter, κ, is related to the constant bulk permeability, π, by the following relation: $\kappa = \pi/(n_0\rho_0^{FR} g_{\text{earth}})$. For instance, for $\pi = 10^7\,\text{kg/m}^3\,\text{s}$ is the hydraulic conductivity $\kappa = 0.1$ darcy. The first three constants are classical elasticity coefficients of the skeleton and of the fluid, Q is the coupling constant introduced by M. A. Biot and β, N are coupling parameters which we investigate in this paper. The transport coefficient of porosity Φ_0 and the parameter δ of equilibrium changes of porosity were introduced, for instance, in the work [15] and one can find in this work their estimates for granular materials. τ is the relaxation time of porosity. As it is clear from (2.9) for undeformed states, $e = 0$, the porosity is equal to its initial value n_0.

Field equations in the explicit form are as follows:

$$\frac{\partial \varepsilon}{\partial t} = \text{ div } \mathbf{v}^F, \qquad \frac{\partial \mathbf{e}^S}{\partial t} = \text{ sym grad } \mathbf{v}^S,$$

$$\begin{aligned} \rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = {} & \text{grad}\left\{\left(\lambda^S - \beta n_0 \delta\right) e + Q\varepsilon + \beta\left(n - n_0\right)\right\} + \text{ div } \left\{2\mu^S \mathbf{e}^S\right\} \\ & + \int\limits_{-\infty}^{t} \pi\left(t-s\right)\cdot\left(\mathbf{v}^F - \mathbf{v}^S\right) ds - N \text{grad}\left(n - n_0\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = {} & \text{grad}\left\{\rho_0^F \kappa\varepsilon + \left(Q + \beta n_0\delta\right) e - \beta\left(n - n_0\right)\right\} \\ & - \int\limits_{-\infty}^{t} \pi\left(t-s\right)\cdot\left(\mathbf{v}^F - \mathbf{v}^S\right) ds + N \text{ grad } \left(n - n_0\right), \end{aligned}$$

$$\frac{\partial\left(n - n_0\right)}{\partial t} - n_0\delta\frac{\partial e}{\partial t} + \Phi_0 \text{ div}\left(\mathbf{v}^F - \mathbf{v}^S\right) = -\frac{n - n_0}{\tau} + \frac{n_0\delta}{\tau} e.$$

Further in this paper we investigate this system under the assumption of a plane wave solution.

3. General propagation condition of monochromatic waves

We assume that the fields of the model satisfy the following relations:

$$
\begin{aligned}
&\varepsilon = E^F\mathcal{E}, \quad \mathbf{e}^S = \mathbf{E}^S\mathcal{E}, \quad \mathbf{v}^F = \mathbf{V}^F\mathcal{E}, \quad \mathbf{v}^S = \mathbf{V}^S\mathcal{E}, \quad n - n_0 = D\mathcal{E},\\
&\mathcal{E} := \exp i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right),
\end{aligned}
\tag{3.1}
$$

where $E^F, \mathbf{E}^S, \mathbf{V}^F, \mathbf{V}^S, D$ are constant amplitudes, ω is a given frequency, $\mathbf{k}$ is the, possibly complex, wave vector. This means that $\mathbf{k} = k\mathbf{n}$, where k is the complex wave number and $\mathbf{n}$ is a unit vector in the direction of propagation. Such a solution describes the propagation of plane monochromatic waves in an infinite medium whose fronts are perpendicular to $\mathbf{n}$.

Substitution of the above relations in field Eqs. $(2.11)_{1,2}$ yields the following compatibility relations:

$$
\begin{aligned}
&E^F = -\frac{1}{\omega}k\mathbf{n}\cdot\mathbf{V}^F, \qquad \mathbf{E}^S = -\frac{1}{2\omega}k\left(\mathbf{n}\otimes\mathbf{V}^S + \mathbf{V}^{\mathbf{S}}\otimes\mathbf{n}\right),\\
&\text{i.e.}\quad e = -\frac{1}{\omega}k\mathbf{n}\cdot\mathbf{V}^S\mathcal{E},
\end{aligned}
\tag{3.2}
$$

and the porosity balance Eq. $(2.11)_5$ implies

$$
D = -\frac{i\Phi_0}{-i\omega + 1/\tau}k\mathbf{n}\cdot\left(\mathbf{V}^F - \mathbf{V}^S\right) - \frac{n_0\delta}{\omega}k\mathbf{n}\cdot\mathbf{V}^S. \tag{3.3}
$$

Making use of these relations in the remaining field equations leads to the following set:

$$
\begin{aligned}
\omega^2\mathbf{V}^S = {}& \frac{\lambda^S - n_0\beta\delta}{\rho_0^S}k^2\left(\mathbf{V}^S\cdot\mathbf{n}\right)\mathbf{n} + \frac{\mu^S}{\rho_0^S}k^2\left(\left(\mathbf{V}^S\cdot\mathbf{n}\right)\mathbf{n} + \mathbf{V}^S\right)\\
&+ \frac{Q}{\rho_0^S}k^2\left(\mathbf{V}^F\cdot\mathbf{n}\right)\mathbf{n} + \frac{N-\beta}{\rho_0^S}\left\{\frac{\omega^2\Phi_0}{\omega^2 + 1/\tau^2}k^2\left(\mathbf{V}^F\cdot\mathbf{n} - \mathbf{V}^S\cdot\mathbf{n}\right) - n_0\delta k^2\mathbf{V}^S\cdot\mathbf{n}\right\}\mathbf{n}\\
&+ i\left\{\frac{\pi^*(\omega)\,\omega}{\rho_0^S}\left(\mathbf{V}^F - \mathbf{V}^S\right) - \frac{N-\beta}{\rho_0^S}\frac{\Phi_0\omega}{\tau\left(\omega^2 + 1/\tau^2\right)}k^2\left(\mathbf{V}^F\cdot\mathbf{n} - \mathbf{V}^S\cdot\mathbf{n}\right)\mathbf{n}\right\} = 0,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\omega^2\mathbf{V}^F = {}& \kappa k^2\left(\mathbf{V}^F\cdot\mathbf{n}\right)\mathbf{n} + \frac{Q + n_0\beta\delta}{\rho_0^F}k^2\left(\mathbf{V}^S\cdot\mathbf{n}\right)\mathbf{n}\\
&- \frac{N-\beta}{\rho_0^F}k^2\left\{\frac{\omega^2\Phi_0}{\omega^2 + 1/\tau^2}\left(\mathbf{V}^F\cdot\mathbf{n} - \mathbf{V}^S\cdot\mathbf{n}\right) - n_0\delta\mathbf{V}^S\cdot\mathbf{n}\right\}\mathbf{n}\\
&- i\left\{\frac{\pi^*(\omega)\,\omega}{\rho_0^S}\left(\mathbf{V}^F - \mathbf{V}^S\right) - \frac{N-\beta}{\rho_0^F}\frac{\Phi_0\omega}{\tau\left(\omega^2 + 1/\tau^2\right)}k^2\left(\mathbf{V}^F\cdot\mathbf{n} - \mathbf{V}^S\cdot\mathbf{n}\right)\mathbf{n}\right\} = 0
\end{aligned}
\tag{3.5}
$$

where

$$\pi^*(\omega) := \int_0^\infty \pi(\eta)\, e^{i\omega\eta} d\eta. \tag{3.6}$$

The complex permeability coefficient $\pi^*(\omega)$ has an extensive literature within the Biot model but it is of no particular interest for our present study. We limit our attention to a particular case in which $\pi^*(\omega)$ is a real constant. This corresponds to the case when the convolution integral in the relation (2.8) reduces to the product $\pi\left(\mathbf{v}^F - \mathbf{v}^S\right)$. In this case, we must replace $\pi^*(\omega)$ by π in the above relations.

It is convenient to separate contributions of the normal and transversal components of the wave vector $k\mathbf{n}$. Let us begin with the transversal component. We take the scalar product of the above equations with an arbitrary unit vector $\mathbf{n}_\perp$ perpendicular to $\mathbf{n}$, i.e. $\mathbf{n} \cdot \mathbf{n}_\perp = 0$. We obtain

$$\begin{gathered}
\omega^2 V_\perp^S = \frac{\mu^S}{\rho_0^S} k^2 V_\perp^S + i \frac{\pi\omega}{\rho_0^S}\left(V_\perp^F - V_\perp^S\right), \\
V_\perp^S := \mathbf{V}^S \cdot \mathbf{n}_\perp, \qquad V_\perp^F := \mathbf{V}^F \cdot \mathbf{n}_\perp, \\
\omega^2 V_\perp^F = -i \frac{\pi\omega}{\rho_0^F}\left(V_\perp^F - V_\perp^S\right).
\end{gathered} \tag{3.7}$$

The dispersion relation follows in the form

$$\omega\left(1 - \frac{\mu^S}{\rho_0^S}\left(\frac{k}{\omega}\right)^2\right) + i\pi \frac{\rho_0^F + \rho_0^S}{\rho_0^F \rho_0^S}\left(1 - \frac{\mu^S}{\rho_0^F + \rho_0^S}\left(\frac{k}{\omega}\right)^2\right) = 0. \tag{3.8}$$

Obviously, this result for shear (transversal) waves is independent of all the interesting parameters: Q, N and β, as could be expected from the structure of constitutive relations in which they contribute only through volume changes of components. For the phase speeds in two limits of frequencies we obtain the well-known results (e.g. [6, 11])

$$\begin{aligned}
\omega \to \infty \quad &\Longrightarrow \quad c_{ph} = \sqrt{\frac{\mu^S}{\rho_0^S}}, \\
\omega \to 0 \quad &\Longrightarrow \quad c_{ph} = \sqrt{\frac{\mu^S}{\rho_0^F + \rho_0^S}}.
\end{aligned} \tag{3.9}$$

Further we limit our attention only to the contribution of the normal component.

4. Longitudinal waves

Let us take the scalar product of Eqs. (3.4) with the vector $\mathbf{n}$. We obtain

$$\begin{aligned}&\left\{\omega^2-\frac{\lambda^S+2\mu^S-n_0\beta\delta}{\rho_0^S}k^2+\frac{N-\beta}{\rho_0^S}\left(\frac{\omega^2\Phi_0}{\omega^2+1/\tau^2}+n_0\delta\right)k^2\right.\\&\qquad\left.+i\left(\frac{\pi\omega}{\rho_0^S}-\frac{N-\beta}{\rho_0^S}\frac{\omega\Phi_0}{\tau\left(\omega^2+1/\tau^2\right)}k^2\right)\right\}V_\parallel^S\\&+\left\{-\frac{Q}{\rho_0^S}k^2-\frac{N-\beta}{\rho_0^S}\frac{\omega^2\Phi_0}{\omega^2+1/\tau^2}k^2-i\left(\frac{\pi\omega}{\rho_0^S}-\frac{N-\beta}{\rho_0^S}\frac{\omega\Phi_0}{\tau\left(\omega^2+1/\tau^2\right)}k^2\right)\right\}V_\parallel^F=0,\end{aligned}$$

(4.1)

$$\begin{aligned}&\left\{-\frac{Q+n_0\beta\delta}{\rho_0^F}k^2-\frac{N-\beta}{\rho_0^F}\left(\frac{\omega^2\Phi_0}{\omega^2+1/\tau^2}+n_0\delta\right)k^2\right.\\&\qquad\left.-i\left(\frac{\pi\omega}{\rho_0^F}-\frac{N-\beta}{\rho_0^F}\frac{\omega\Phi_0}{\tau\left(\omega^2+1/\tau^2\right)}k^2\right)\right\}V_\parallel^S\\&+\left\{\omega^2-\kappa k^2+\frac{N-\beta}{\rho_0^F}\frac{\omega^2\Phi_0}{\omega^2+1/\tau^2}k^2+i\left(\frac{\pi\omega}{\rho_0^F}-\frac{N-\beta}{\rho_0^F}\frac{\omega\Phi_0}{\tau\left(\omega^2+1/\tau^2\right)}k^2\right)\right\}V_\parallel^F=0,\end{aligned}$$

where

$$V_\parallel^S=\mathbf{V}^S\cdot\mathbf{n},\qquad V_\parallel^F=\mathbf{V}^F\cdot\mathbf{n}. \tag{4.2}$$

Before we proceed to the numerical analysis of the above eigenvalue problem we demonstrate two special cases.

The simplest case follows for the so-called simple mixture model defined by the relations

$$\beta=0,\qquad N=0,\qquad Q=0,\qquad \tau\to\infty. \tag{4.3}$$

Then the set (4.1) reduces to the form

$$\begin{aligned}&\left(\omega^2-\frac{\lambda^S+2\mu^S}{\rho_0^S}k^2+i\frac{\pi\omega}{\rho_0^S}\right)V_\parallel^S-i\frac{\pi\omega}{\rho_0^S}V_\parallel^F=0,\\&-i\frac{\pi\omega}{\rho_0^F}V_\parallel^S+\left(\omega^2-\kappa k^2+i\frac{\pi\omega}{\rho_0^F}\right)V_\parallel^F=0.\end{aligned} \tag{4.4}$$

This yields the dispersion relation

$$\omega\left(1-\frac{\lambda^S+2\mu^S}{\rho_0^S}\left(\frac{k}{\omega}\right)^2\right)\left(1-\kappa\left(\frac{k}{\omega}\right)^2\right) + i\pi\frac{\rho_0^F+\rho_0^S}{\rho_0^F\rho_0^S}\left(1-\frac{\lambda^S+2\mu^S+\rho_0^F\kappa}{\rho_0^F+\rho_0^S}\left(\frac{k}{\omega}\right)^2\right)=0. \tag{4.5}$$

This relation was extensively discussed in earlier works (for quotations see, e.g. [6, 11]). It yields the existence of two longitudinal modes of propagation: P1- and P2-waves. In the frequency limits we obtain easily for the phase velocities

$$\begin{aligned}
&\omega\to\infty \quad\Longrightarrow\quad c_{ph}=\begin{cases}\sqrt{\dfrac{\lambda^S+2\mu^S}{\rho_0^S}} & \text{for P1-waves,}\\ \sqrt{\kappa} & \text{for P2-waves,}\end{cases}\\
&\omega\to 0 \quad\Longrightarrow\quad c_{ph}=\begin{cases}\sqrt{\dfrac{\lambda^S+2\mu^S+\rho_0^F\kappa}{\rho_0^F+\rho_0^S}} & \text{for P1-waves,}\\ 0 & \text{for P2-waves.}\end{cases}
\end{aligned} \tag{4.6}$$

The second particular case follows under the following assumptions

$$\beta=0,\qquad N=0,\qquad \tau\to\infty. \tag{4.7}$$

Then we obtain the following dispersion relation describing Biot's model without relative accelerations:

$$\omega\left\{\left(1-\frac{\lambda^S+2\mu^S}{\rho_0^S}\left(\frac{k}{\omega}\right)^2\right)\left(1-\kappa\left(\frac{k}{\omega}\right)^2\right)-\frac{Q^2}{\rho_0^F\rho_0^S}\left(\frac{k}{\omega}\right)^4\right\} + i\pi\frac{\rho_0^F+\rho_0^S}{\rho_0^F\rho_0^S}\left(1-\frac{\lambda^S+2\mu^S+\rho_0^F\kappa+2Q}{\rho_0^F+\rho_0^S}\left(\frac{k}{\omega}\right)^2\right)=0. \tag{4.8}$$

Again there is an extensive literature on this equation. In the limit frequencies we have

$$\omega\to\infty \quad\Longrightarrow\quad c_{ph}=\begin{cases}\dfrac{\sqrt{2}}{D^+}\sqrt{\dfrac{\lambda^S+2\mu^S}{\rho_0^S}\kappa-\dfrac{Q^2}{\rho_0^S\rho_0^F}} & \text{for}\quad \text{P1-waves,}\\ \dfrac{\sqrt{2}}{D^-}\sqrt{\dfrac{\lambda^S+2\mu^S}{\rho_0^S}\kappa-\dfrac{Q^2}{\rho_0^S\rho_0^F}} & \text{for}\quad \text{P2-waves.}\end{cases} \tag{4.9}$$

(4.9) [cont.]

$$D^{\pm}: \quad = \sqrt{\frac{\lambda^S + 2\mu^S}{\rho_0^S} + \kappa \mp \sqrt{\left(\frac{\lambda^S + 2\mu^S}{\rho_0^S} - \kappa\right)^2 + \frac{4Q^2}{\rho_0^S \rho_0^F}}},$$

$$\omega \to 0 \quad \Longrightarrow \quad c_{ph} = \begin{cases} \sqrt{\dfrac{\lambda^S + 2\mu^S + \rho_0^F \kappa + 2Q}{\rho_0^F + \rho_0^S}} & \text{for P1-waves,} \\ 0 & \text{for P2-waves.} \end{cases}$$

We shall not discuss these relations in any details and proceed to the numerical analysis of the general relation (4.1).

5. Numerical analysis of the dispersion relation

As already mentioned, the propagation of monochromatic shear waves is independent of parameters β, Q, N which are of the main interest in this work. For this reason we concentrate on longitudinal waves.

We investigate the dispersion relation (4.1) for the following numerical data corresponding approximately to, for instance, either marls or porous and saturated sandstones [4]

$$\begin{aligned} \rho_0^S &= 2500 \text{ kg/m}^3, \qquad \rho_0^F = 250 \text{ kg/m}^3, \\ n_0 &= 0.25, \qquad \pi = 10^7 \text{ kg/m}^3\text{s}, \qquad \tau = 10^{-3} \text{ s}, \\ \lambda^S &= 7 \cdot 10^9 \text{ N/m}^2, \qquad \mu^S = 4.3 \cdot 10^9 \text{ N/m}^2, \qquad \kappa = 2.25 \cdot 10^6 \text{ m}^2/\text{s}^2, \\ \delta &= 3, \qquad \Phi = 0.06 \cdot n_0 = 0.015. \end{aligned}$$

These data lead to the following limit values of the speeds of propagation for shear waves:

$$\lim_{\omega \to \infty} c_{ph} = c_S = \sqrt{\frac{\mu_S}{\rho_0^S}} = 1311 \text{ m/s},$$

and for P1- and P2-waves modelled by the simple mixture model (i.e. for $\beta = 0, N = 0, Q = 0$)

$$\lim_{\omega \to \infty} c_{ph} = \begin{cases} c_{P1} = \sqrt{\dfrac{\lambda^S + 2\mu^S}{\rho_0^S}} = 2498 \text{ m/s}, \\ c_{P2} = \sqrt{\kappa} = 1500 \text{ m/s}, \end{cases}$$

which are almost identical with the data which we have been used in previous works on this subject (e.g.: [2, 6, 7]). The values of parameters δ, Φ_0 follow from the estimates made by means of modified Gassmann relations [15].

In Fig. 1, we present results for the P1-wave and for the P2-wave without any influence of β (i.e. $\beta = 0$). The choice of values for Q, N is again motivated by results following from the modified Gassmann relation [15]. They are of the same order of magnitude and, in the example, have been chosen to be either zero or 0.25 GPa. While the choice of the values for Q, N is nearly without the influence in the case of the P2-wave, as we see in the right panel of Fig. 1, for the P1-wave their values are of importance. The coupling parameter Q shifts the values of the speed of propagation to higher values and flattens the curve. This property has been already indicated in the works [6] and [16]. The curves remain monotonic with respect to the frequency. This is not the case any more if we choose N unequal to zero. The speed has in these cases a well pronounced minimum in the range of medium frequencies of some kilohertz. Such a minimum for surface waves described by Biot's model was discovered by BOURBIE, COUSSY and ZINSZNER [4] and it has been confirmed by B. Albers for surface waves described by the simple mixture model [3]. It was attributed to the coupling between P1- and P2-waves. It seems that for bulk waves the coupling must be amplified by porosity changes in order to be visible.

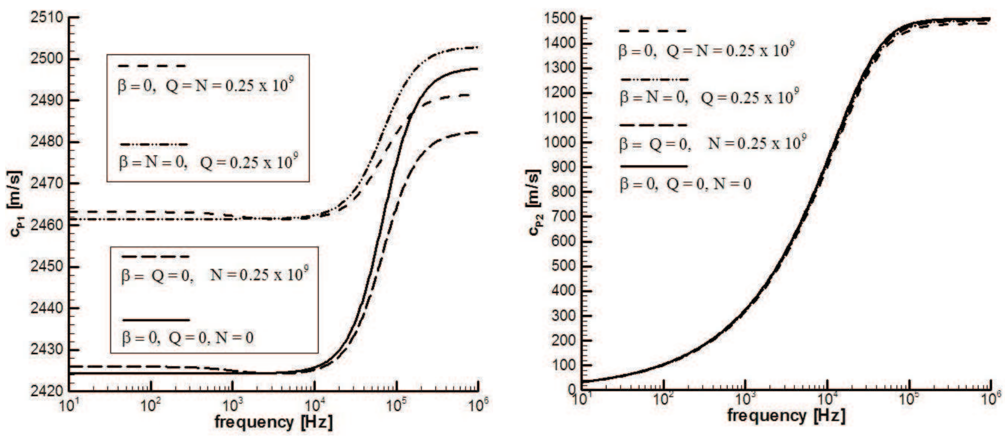

FIG. 1. Speeds of monochromatic P1-waves (left) and P2-waves (right) for various combinations of values of parameters Q, N and for $\beta = 0$.

We proceed to investigate the influence of the parameter β responsible for nonequilibrium changes of porosity. There exist neither experimental evidence nor theoretical estimates of its values. As it appears in the model in an additive combination with the parameter N or, multiplied by $n_0\delta$ which is of the order of unity, with parameters Q, λ^S, μ^S (compare (4.1)), we expect its values to have at most the same order of magnitude as these constants. Consequently, we should concentrate on values of $\beta \leq 10^9$ Pa. However, in order to show that higher values yield physically unreasonable results for the wave speeds, we present in

Fig. 2 also some values of $\beta > 10^9$. This figure shows both the speeds of the P1-wave and of the P2-wave for nine values of β ($\beta = 0,\ 10^5 \le \beta \le 10^{12}$). The P1-wave is presented in solid lines, the P2-waves in dashed lines. The frequency is plotted in a logarithmic scale.

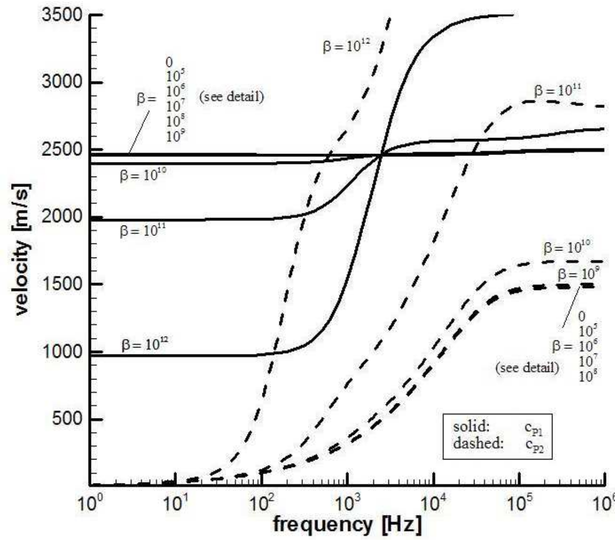

FIG. 2. Speeds of P1-waves (solid) and P2-waves (dashed) for various values of β. The frequency is plotted in a logarithmic scale.

It is clearly seen that both for the P1- and for the P2-wave this parameter in the range $0 \le \beta \le 10^9$ yields no considerable changes of the speeds, i.e. β has no qualitative influence on the results and it yields small quantitative changes. This turns differently for values larger than $\beta = 10^9$. For those values the influence of the parameter is immense but the speeds of the waves are not physically reasonable anymore. Their limit values for $\omega \to \infty$ become larger than speeds of longitudinal waves in pure substances: in the solid for P1-waves, and in the fluid for P2-waves. However, as described above, we had to expect this result because in these cases β was much larger than comparable parameters in the dispersion equation.

We do not further deal with such values but show in Fig. 3 in detail the behavior of the waves for reasonable values of β, i.e. $0 \le \beta \le 10^9$. In this figure we see the speeds of both waves in a normal scale. Especially from the details in both panels which show the speeds for very large frequencies and a zoom of the speed axis, it is visible that at least for values $\beta \le 10^8$ there exists no influence of this parameter at all. Also the influence of $\beta = 10^9$ is tiny and it is justified – not only in order to reduce technical difficulties of numerical calculations – to neglect in linear models the influence of the coupling parameter completely and to use the value $\beta = 0$.

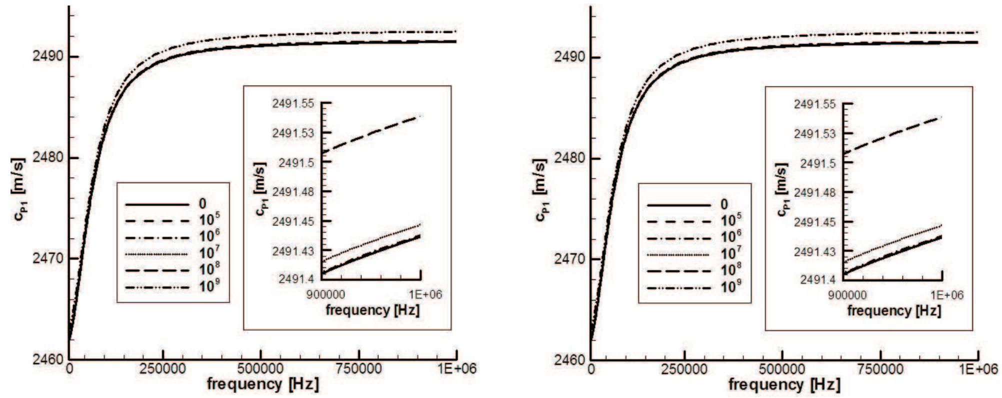

Fig. 3. Speeds of P1-waves (left) and P2-waves (right) for physically reasonable values of β. The frequency is plotted in a normal scale. The details are incorporated to show the small differences between relatively small values of β.

6. Conclusions

The results presented above support the assumption which we made in earlier works on the propagation of linear acoustic waves in saturated porous materials, that the nonequilibrium coupling through porosity changes described by the parameter β can be neglected. As a preliminary analysis of nonlinear waves indicates [8], this may not be the case for nonlinear waves where it yields the existence of solitary waves of porosity.

On the other hand, the influence of the porosity gradient described by the constant N is nontrivial. In the range of medium frequencies, it creates a minimum in the phase speed of P1-waves. This yields considerable changes in group speeds which we do not demonstrate in this work. Consequently, one may expect a creation of the Airy phase (see: [1] for the analysis of this problem in modelling of single component media). We will return to this problem in the forthcoming publication.

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