ARCHIVES OF MECHANICS
Arch. Mech. **77** (3), 217–262, 2025, https://doi.org/10.24423/aom.4564

Improved three-dimensional refined plate theory

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The paper presents improvements over an earlier developed three-dimensional refined plate theory. The improved theory removes the disadvantage of the earlier theory in that it does not properly satisfy transverse shear stress conditions, and deficiency in being suitable only for flexure problems. The improved theory is suitable for use in flexure, as well as, for vibrations and stability problems of plates. The theory is simple, easy to use and accurate. The number of unknown variables involved are the same as those associated with thin plates, *viz.* only one in the case of flexure and vibrations; and three in the case of stability. The theory is based on displacement. The theory, to keep it as simple as possible, uses the concept of targeted displacements (which contribute only towards specific stresses, moments, shear forces, axial forces). All the stresses are represented realistically. The theory uses all strain displacement relations, and satisfies, as accurately as possible, all constitutive relations. The moments and forces satisfy gross equilibrium equations. The theory has some noteworthy similarities with the earlier developed well known theories. Due to these similarities, the experience of dealing with the earlier developed theories can be harnessed. Illustrative examples bring out the eficacy of the theory.

Key words: plate theory, thick plate theory, exact plate theory, three-dimensional plate theory, refined plate theory.

1. Introduction

Plates are important elements of many civil, mechanical, aerospace structures.

Classical Plate Theory (CPT), also known as Love–Kirchhoff plate theory and also as thin plate theory, was developed in the late nineteenth century. The CPT is widely in use even today because of its simplicity. A book by Timoshenko and Woinowsky-Krieger [1] is the most authentic text on the CPT.

The CPT takes into account only in-plane stresses and does not take into account effects of transverse stresses.

In 1945, Reissner [2] presented a stress based plate theory; and subsequently in 1951, Mindlin [3] presented a displacement based plate theory, which took into account transverse shear stresses. Many theories got developed afterwards.

Important amongst these are theories by Lo *et al.* [4], Krishna Murty [5], Levinson [6], Kant [7], Reddy [8], Carerra [9], Shimpi [10]. Most of these theories are based on displacement. Exact elasticity theories were given by Srinivas *et al.* [11], Pagano [12]. Review papers [13–15] give a good account of the developments in the plate theory.

Plate theories are constantly evolving. There is always a need for simple yet accurate theories.

1.1. Desirability for a plate theory to have minimum number of unknown variables

There are some important observations about development of plate theories:

1) It was noted by Lo *et al.* [4, p. 664]:

> "Plate theories can be developed by expanding the displacements in power series of the coordinate normal to the middle plane. In principle, theories developed by this means can be made as accurate as desired simply by including a sufficient number of terms. In practice, however, a point of diminishing returns is reached whereby the complexity of the resulting forms becomes too great."

2) It was noted by Abrate and Di Sciuva [15, p. 490]:

> "Displacement approximations can be expressed as polynomial or non-polynomial expansions in terms of the transverse coordinate. It is expected that the number of terms retained increases, the accuracy of the predicting will improve. However the number of unknowns to be determined will increase, leading to a more complex formulation."

As the number of unknown variables increase, complications also increase, also identifying correct specifications of boundary conditions becomes increasingly difficult.

Therefore, it is always desirable to have a fairly accurate theory having less number of variables.

1.2. Developments of fairly accurate theories having two or less unknown variables

Two variable theories: In 2002, in [10], the refined plate theory (RPT) and its variants were presented. The RPT has received a fairly good response in the literature. The zeroth-order shear deformation theory for plates [16] was a precursor to the RPT.

The RPT had two variables, it was a variationally consistent shear deformation theory, wherein transverse shear stresses across thickness were parabolic in nature satisfying zero shear stress conditions on plate surfaces. The theory gave very accurate results; one of the governing equations and boundary conditions had striking similarities with those of the CPT.

The RPT was further extended for vibrations [17], orthotropic plates [18]. Also, the RPT was the base for two new first-order shear deformation plate theories [19].

Single variable theories: One of the variants of the RPT was RPT-Variant II [10]. The RPT-Variant II shared many characteristics of the RPT, but it involved only one variable, it was variationally inconsistent, and had the governing equation and boundary conditions strikingly similar to those of the CPT. And, surprisingly, the results obtained using the RPT and the RPT-Variant II were almost identical.

The RPT-Variant II [10] was used as a base in plate vibrations [20], pull-in instability of microbeams [21], three-dimensional refined plate theory using targeted displacements [22].

1.3. Concept of 'targeted displacement'

To the best knowledge of the author, the term 'targeted displacement' was first used in [22]. The term was used in respect of some components of displacements, which contribute only towards specific moments, shear forces, and stresses.

Now, from the hindsight, one can say that even the CPT inherentely used the targeted displacement concept. The displacements of the CPT give rise to only flexural strains and flexural stresses, as a result, give rise to only moments. The transverse shear strains in the CPT are identically zero. Therefore, if constitutive relations are used to obtain stresses, transverse shear stresses and transverse shear forces would turn out to be identically zero. Therefore, the displacements of the CPT can be considered to be targeted displacements, which contribute only towards moments.

The appropriate use of targeted displacements in a theory can result in reduction of complexity as was shown in [22].

1.4. Aim of the present work

In [22], "Three-dimensional refined plate theory using targeted displacements and its variant" was presented. The theory, though otherwise good, has the following deficiencies:

1) The transverse shear stress boundary conditions were not properly satisfied.
2) The theory was suitable only for flexure problems of a plate.

With the aim to remove the deficiencies of [22], the present paper "Improved three-dimensional refined plate theory" (Improved 3D-RPT) free of these disadvantages is presented.

The Improved 3D-RPT is suitable for use in flexure, as well as, in vibrations and stability problems of plates. In the theory, displacement definitions include inertial terms. Effects of all the stresses are taken into account. The theory is easy to use and gives accurate results for thin, as well as, thick plates.

The scope would include flexure, free vibrations, static stability.

About some terminologies used in this paper: the term 'flexure' used here is synonymous with the term 'bending'; and, the term 'vibrations' used here would inherently mean 'free vibrations'.

2. The distinguishing differences between the present work and the earlier work

The present work is an improvement over the earlier work [22], it is necessary to point out the major differences between them.

Seemingly, it may appear that the present work is a straightforward extension of [22]*, but it is not the case for the following reasons:*

1) The displacements of [22] do not contain inertial terms.
   Whereas, the displacements of the present work contain inertial terms.
2) In [22], the number of targeted displacement terms is 7.
   Whereas, in the present work, the number of targeted displacement terms is 17.
3) The transverse shear stresses of [22] were not completely satisfying shear stress free boundary conditions on the surfaces of the plate. This deficiency has been removed in the present work.
4) In the case of plate vibrations, if the displacement expressions of [22] are used, expressions for transverse shear forces would result in inconsistencies. In support, it can be noted that expressions for transverse shear forces can be obtained in two ways:

   (a) *Direct way*: using strain-displacement relations, transverse shear strains can be obtained; and then using appropriate constitutive relations, transverse shear stresses can be obtained. Then using these stresses, expressions for transverse shear forces can be obtained in a straightforward way.

   (b) *Indirect way*: using strain-displacement relations and constitutive relations, one can obtain flexural stresses; and using these stresses, moments can be obtained.

Now, how to obtain expressions for shear forces in the case of the CPT for a static problem, is described in [1, pp. 81–82]. Following similar procedure and by using gross equilibrium equations for plate vibrations, expressions for shear forces can be obtained in the case of plate vibrations in an indirect tedious way.

If displacements of [22] are used, then the shear force expressions obtained by procedures given in the just mentioned items (a) and (b) would be different from each other (one would not have inertial terms, whereas the other would have inertial terms) and this is the inconsistency.

But, such inconsistency would not be there in the present work, because the displacements of the present work are substantially different from those of [22] and contain appropriate inertial terms.

5) In [22], there is no provision for in-plane forces. The present formulation removes this deficiency.

Therefore, it can be seen that the present work is not a straightforward extension of work reported in [22]. But, the present work is substantially different from that of [22] and has its own distinct and unique features.

3. Plate under consideration

1) *Geometry of the plate*: A right-handed Cartesian coordinate system o-x-y-z would be utilized.
   - (a) The plate is of uniform thickness h.
   - (b) The mid-surface of the unloaded plate lies in xy-plane.
   - (c) The origin o of the coordinate system o-x-y-z is chosen at a convenient location.
   - (d) The present work is valid for a plate having any plan form.
   - (e) For the sake of convenience of derivation and for illustration, a rectangular plate (of length a, width b, thickness h) is considered. The unloaded plate occupies the region:

$$0 \le x \le a, \quad 0 \le y \le b, \quad -h/2 \le z \le h/2. \tag{3.1}$$

The term 'plate surfaces' would be used to indicate plate surfaces at $z = \pm h/2$.

2) *Loading on the plate*:
   - (a) In the case of plane stress:
     The plate is subjected only to static in-plane forces N_x, N_y, N_{xy} and applied at the mid-surface of the plate.

The surfaces $z = \pm h/2$ do not have lateral loading and are shear-stress free.

(b) In the case of flexure:
The plate is loaded only by a static lateral load of intensity $q(x, y)$ on surface $z = -h/2$. The lateral loading would be considered positive, if it is acting along the positive z-direction. The surfaces $z = \pm h/2$ are shear-stress free. There is no in-plane loading.

(c) In the case free vibrations:
There is no in-plane loading, as well as, there is no lateral loading.
The surfaces $z = \pm h/2$ are shear-stress free.

(d) In the case of static stability of plate:
The loading would be a combination of the case of plane stress and of the case of flexure (i.e., a combination of just mentioned cases in items (a) and (b)).

3) *Material of the plate*:

(a) The plate is made of linearly elastic, homogenious, isotropic material.
(b) The modulus of elasticity of the material is E.
(c) Poisson's ratio is μ.
(d) The shear modulus is G, where $G = E/[2(1+\mu)]$.
(e) The density of the material is ρ.

4. Some theory of elasticity relations, definitions of moments, forces

The Timoshenko notation [23] would be used for strains, stresses.

1) *Strain definitions*: Strains are related to displacements u, v, w (in x-, y-, z-directions, respectively) as follows:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x}, & \epsilon_y &= \frac{\partial v}{\partial y}, & \epsilon_z &= \frac{\partial w}{\partial z}, \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, & \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, & \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \end{aligned} \tag{4.1}$$

2) *Constitutive relations of the theory of elasticity*: Strains are expressed in terms of stresses by following constitutive relations:

$$\epsilon_x = \big[\sigma_x - \mu\sigma_y - \mu\sigma_z\big]/E, \tag{4.2}$$

$$\epsilon_y = \big[\sigma_y - \mu\sigma_z - \mu\sigma_x\big]/E, \tag{4.3}$$

$$\epsilon_z = \big[\sigma_z - \mu\sigma_x - \mu\sigma_y\big]/E, \tag{4.4}$$

$$\gamma_{xy} = \tau_{xy}\big[2(1+\mu)/E\big], \tag{4.5}$$

$$\gamma_{yz} = \tau_{yz}\big[2(1+\mu)/E\big], \tag{4.6}$$

$$\gamma_{zx} = \tau_{zx}\big[2(1+\mu)/E\big]. \tag{4.7}$$

3) *Definitions of in-plane forces, moments, shear forces*:
In-plane forces (N_x, N_y, N_{xy}), moments (M_x, M_y, M_{xy}), shear forces (Q_x, Q_y) are defined as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \int\limits_{z=-h/2}^{z=h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \sigma_x\ z \\ \sigma_y\ z \\ \tau_{xy}\ z \\ \tau_{zx} \\ \tau_{yz} \end{Bmatrix} dz. \tag{4.8}$$

5. Assumptions

Some appropriate assumptions would be made:

1) It would be assumed that the displacements involved are small such that strain-displacement relations, as given by Eq. (4.1), of the theory of elasticity hold good.
2) In the plane stress case and in the plate vibration case: as there is no lateral loading, and as transverse strains are not prevented, transverse normal stress σ_z would hardly be there; and it can be safely assumed that: $\sigma_z = 0$. Using constitutive relations (4.2) and (4.3), one then gets:

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y) \quad \text{and} \quad \sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x).$$

3) In the plate flexure case: there is lateral loading, but transverse strains are not prevented.
The transverse normal stress σ_z is, in general, very small compared to in-plane stresses σ_x and σ_y.
In constitutive relation (4.2), there are terms σ_x, $\mu\sigma_y$ and $\mu\sigma_z$.
In constitutive relation (4.3), there are terms σ_y, $\mu\sigma_x$ and $\mu\sigma_x$.
Therefore, in constitutive relations (4.2), (4.3), it is justifiable to ignore the term $\mu\sigma_z$ (as σ_z itself is, in general, very small compared to in-plane stresses σ_x and σ_y).

Using constitutive relations (4.2) and (4.3), one then gets:

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y) \quad \text{and} \quad \sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x).$$

4) In the constitutive relation (4.4), the terms involved are σ_z, and $\mu\sigma_x$, $\mu\sigma_y$. No term needs to be ignored. The constitutive relation (4.4) should be satisfied as accurately as possible; and while satisfying constitutive relation (4.4), ignorable higher-order entities in the stresses σ_x and σ_y can be safely ignored. It needs to be noted that the constitutive relation (4.4) is completely ignored in many theories (e.g., the CPT [1], Mindlin's theory [3], the RPT [10]).
5) The constitutive relations (4.5)–(4.7), which relate shear strains γ_{xy}, γ_{yz}, γ_{zx} to shear stresses τ_{xy}, τ_{yz}, τ_{zx}, respectively, need to be satisfied.
6) In the case of static stability of the plate, the in-plane forces and lateral loading both can be there.
   (a) When only in-plane forces are applied to the plate, there would be in-plane stresses in the plate, and these in-plane stresses would remain practically the same even when additional lateral loading is also applied.
   (b) When only a lateral load is applied to the plate, there would be moments and shear forces in the plate, and these moments and shear forces would remain practically the same even when additional in-plane forces are also applied.

These assumption (a) and (b) are in tune with the derivation of the Saint-Venant stability equation (as given in [1, pp. 378–380]).

6. Use of overbraces in this paper

Consider, only just for illustration, the following equation:

$$u = \overbrace{u_o(x,y)}^{u_o} + \overbrace{\left(-z\frac{\partial w_b}{\partial x}\right)}^{u_b} + \overbrace{h^3\phi_1\frac{\partial}{\partial x}(\nabla^2 w_b)}^{u_{c_1}} + \overbrace{h^2\phi_2\frac{1}{E}\frac{\partial q}{\partial x}}^{u_{c_2}} + \overbrace{h^3\phi_3\frac{\rho}{E}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial t^2}\right)}^{u_i}. \tag{6.1}$$

In the preceding equation, five overbraces are used to show that the right hand side of the equation for u has five components. The names of the components (in this case u_o, u_b, u_{c_1}, u_{c_2}, u_i) are written over the corresponding overbraces. *Whatever is written over the overbraces is only for information and explanation purposes.* Wherever necessary, such a convention of using overbraces would be utilized.

7. Displacements

Going by the experience gained from references [10, 16–22] and with some efforts, it is possible to write down the displacements u, v, w (in x-, y-, z-directions, respectively):

$$u = \overbrace{u_o(x,y)}^{u_o} + \overbrace{h^2\phi_1 \frac{\partial}{\partial x}\left(\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}\right)}^{u_{o_c}} + \overbrace{\left(-z\frac{\partial w_b}{\partial x}\right)}^{u_b} + \overbrace{h^3\phi_2 \frac{\partial}{\partial x}(\bigtriangledown^2 w_b)}^{u_{c_1}} + \overbrace{h^2\phi_3 \frac{1}{E}\frac{\partial q}{\partial x}}^{u_{c_2}} + \overbrace{h^3\phi_4 \frac{\rho}{E}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial t^2}\right)}^{u_i}, \tag{7.1}$$

$$v = \overbrace{v_o(x,y)}^{v_o} + \overbrace{h^2\phi_1 \frac{\partial}{\partial y}\left(\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}\right)}^{v_{o_c}} + \overbrace{\left(-z\frac{\partial w_b}{\partial y}\right)}^{v_b} + \overbrace{h^3\phi_2 \frac{\partial}{\partial y}(\bigtriangledown^2 w_b)}^{v_{c_1}} + \overbrace{h^2\phi_3 \frac{1}{E}\frac{\partial q}{\partial y}}^{v_{c_2}} + \overbrace{h^3\phi_4 \frac{\rho}{E}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial t^2}\right)}^{v_i}, \tag{7.2}$$

$$w = \overbrace{-z\phi_5\left(\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}\right)}^{w_{o_c}} + \overbrace{w_b(x,y,t)}^{w_b} + \overbrace{h^2\phi_6(\bigtriangledown^2 w_b)}^{w_{c_1}} + \overbrace{h\phi_7\frac{q}{E}}^{w_{c_2}} + \overbrace{h^2\phi_8 \frac{\rho}{E}\frac{\partial^2 w_b}{\partial t^2}}^{w_i}, \tag{7.3}$$

where u, v, w are functions of coordinates x, y, z and time t,
u_o, v_o are functions of coordinates x, y,
w_{o_c} is a function of coordinates x, y, z,
w_b is a function of coordinates x, y and time t,

$$\phi_1 = \frac{\mu}{2(1-\mu)}\left[\left(\frac{z}{h}\right)^2 - \frac{1}{12}\right], \tag{7.4}$$

$$\phi_2 = \frac{2-\mu}{1-\mu}\left[\frac{1}{6}\left(\frac{z}{h}\right)^3 - \frac{1}{40}\left(\frac{z}{h}\right)\right], \tag{7.5}$$

$$\phi_3 = \left[\frac{1}{10}\left(\frac{z}{h}\right)^5 - \frac{1}{4}\left(\frac{z}{h}\right)^3 + \frac{1}{4}\left(\frac{z}{h}\right)^2 + \frac{39}{1120}\left(\frac{z}{h}\right) - \frac{1}{48}\right], \tag{7.6}$$

$$\phi_4 = (1+\mu)\left[-\frac{1}{3}\left(\frac{z}{h}\right)^3 + \frac{1}{20}\left(\frac{z}{h}\right)\right], \tag{7.7}$$

$$\phi_5 = \frac{\mu}{1-\mu}, \tag{7.8}$$

(7.9) $$\phi_6 = \frac{1}{1-\mu}\left[\frac{\mu}{2}\left(\frac{z}{h}\right)^2 - \frac{8+\mu}{40}\right],$$

(7.10) $$\phi_7 = \left[-\frac{1}{2}\left(\frac{z}{h}\right)^4 + \frac{3}{4}\left(\frac{z}{h}\right)^2 - \frac{1}{2}\left(\frac{z}{h}\right) - \frac{39}{1120}\right],$$

(7.11) $$\phi_8 = \frac{1+\mu}{5},$$

where the functions would be referred to

u_o, v_o *plane stress* components of displacements u, v, respectively,

u_{o_c}, v_{o_c}, w_{o_c} *complimentary plane stress* components of displacements u, v, w, respectively,

u_b, v_b, w_b *bending* components of displacements u, v, w, respectively (and can be considered to be analogous to the classical plate theory displacements u, v, w, respectively),

u_{c_1}, u_{c_2} *complimentary* components of displacement u,

v_{c_1}, v_{c_2} *complimentary* components of displacement v,

w_{c_1}, w_{c_2} *complimentary* components of displacement w,

u_i, v_i, w_i *inertial* components of displacements u, v, w, respectively.

As the displacements considered are unique and single valued, the compatibility conditions would get automatically satisfied.

8. Obtaining expressions for stresses σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{zx} (using displacements, strain definitions, constitutive relations)

Using expressions for displacements u, v, w (given by expressions (7.1)–(7.3), respectively), using strain definitions (4.1), various strains can be obtained. Using these strains and the constitutive relations (4.2), (4.3), (4.5)–(4.7) and taking into account the assumptions 2 and 3 of Section 5, one gets:

(8.1) $$\sigma_x = \overbrace{\frac{E}{1-\mu^2}\left(\frac{\partial u_o}{\partial x} + \mu\frac{\partial v_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o} + \overbrace{\frac{Eh^2}{1-\mu^2}\phi_1\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}\right)}^{\text{contributions due to } u_{o_c} \text{ and } v_{o_c}}$$
$$+ \overbrace{\left[-\frac{E\,z}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right)\right]}^{\text{contribution due to } u_b \text{ and } v_b} + \overbrace{\frac{Eh^3}{1-\mu^2}\phi_2\left(\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2}\right)(\nabla^2 w_b)}^{\text{contribution due to } u_{c_1} \text{ and } v_{c_1}}$$
$$+ \overbrace{\frac{h^2}{1-\mu^2}\phi_3\left(\frac{\partial^2 q}{\partial x^2} + \mu\frac{\partial^2 q}{\partial y^2}\right)}^{\text{contribution due to } u_{c_2} \text{ and } v_{c_2}} + \overbrace{\frac{\rho h^3}{1-\mu^2}\phi_4\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right)}^{\text{contribution due to } u_i \text{ and } v_i},$$

(8.2)
$$\sigma_y = \overbrace{\frac{E}{1-\mu^2}\left(\frac{\partial v_o}{\partial y}+\mu\frac{\partial u_o}{\partial x}\right)}^{\text{contribution due to } u_o \text{ and } v_o} + \overbrace{\frac{Eh^2}{1-\mu^2}\phi_1\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}\right)}^{\text{contributions due to } u_{oc} \text{ and } v_{oc}}$$
$$+\overbrace{\left[-\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial y^2}+\mu\frac{\partial^2 w_b}{\partial x^2}\right)\right]}^{\text{contribution due to } u_b \text{ and } v_b} + \overbrace{\frac{Eh^3}{1-\mu^2}\phi_2\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)(\bigtriangledown^2 w_b)}^{\text{contribution due to } u_{c_1} \text{ and } v_{c_1}}$$
$$+\overbrace{\frac{h^2}{1-\mu^2}\phi_3\left(\frac{\partial^2 q}{\partial y^2}+\mu\frac{\partial^2 q}{\partial x^2}\right)}^{\text{contribution due to } u_{c_2} \text{ and } v_{c_2}} + \overbrace{\frac{\rho h^3}{1-\mu^2}\phi_4\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial y^2}+\mu\frac{\partial^2 w_b}{\partial x^2}\right)}^{\text{contribution due to } u_i \text{ and } v_i},$$

(8.3)
$$\tau_{xy} = \overbrace{\frac{E}{2(1+\mu)}\left(\frac{\partial v_o}{\partial x}+\frac{\partial u_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o} + \overbrace{\frac{Eh^2}{1+\mu}\phi_1\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}\right)}^{\text{contribution due to } u_{oc} \text{ and } v_{oc}}$$
$$+\overbrace{\left[-\frac{Ez}{1-\mu^2}(1-\mu)\frac{\partial^2 w_b}{\partial x\partial y}\right]}^{\text{contribution due to } u_b \text{ and } v_b} + \overbrace{\frac{Eh^3}{1+\mu}\phi_2\frac{\partial^2}{\partial x\partial y}(\bigtriangledown^2 w_b)}^{\text{contribution due to } u_{c_1} \text{ and} v_{c_1}}$$
$$+\overbrace{\frac{h^2}{1+\mu}\phi_3\frac{\partial^2 q}{\partial x\partial y}}^{\text{contribution due to } u_{c_2} \text{ and } v_{c_2}} + \overbrace{\frac{\rho h^3}{1+\mu}\phi_4\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x\partial y}\right)}^{\text{contribution due to } u_i \text{ and } v_i},$$

(8.4)
$$\tau_{yz} = \overbrace{\frac{Eh^2}{1-\mu^2}\left[\frac{1}{2}\left(\frac{z}{h}\right)^2-\frac{1}{8}\right]\frac{\partial}{\partial y}(\bigtriangledown^2 w_b)}^{\text{contribution due to } v_{c_1} \text{ and} w_{c_1}}$$
$$+\overbrace{\rho h^2\left[-\frac{1}{2}\left(\frac{z}{h}\right)^2+\frac{1}{8}\right]\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial y}\right)}^{\text{contribution due to } v_i \text{ and } w_i},$$

(8.5)
$$\tau_{zx} = \overbrace{\frac{Eh^2}{1-\mu^2}\left[\frac{1}{2}\left(\frac{z}{h}\right)^2-\frac{1}{8}\right]\frac{\partial}{\partial x}(\bigtriangledown^2 w_b)}^{\text{contribution due to } u_{c_1} \text{ and } w_{c_1}}$$
$$+\overbrace{\rho h^2\left[-\frac{1}{2}\left(\frac{z}{h}\right)^2+\frac{1}{8}\right]\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)}^{\text{contribution due to } u_i \text{ and } w_i}.$$

It should be noted from expressions (8.4) and (8.5) that:

(8.6)
$$[\tau_{yz}]_{z=\pm h/2} = 0 \quad \text{and} \quad [\tau_{zx}]_{z=\pm h/2} = 0.$$

Equation (8.6) indicates that plate surfaces $z = \pm h/2$ are shear stress free.

9. Obtaining of expressions for axial forces, moments, shear forces (using displacements, strain definitions, constitutive relations, definitions for axial forces, moments, shear forces)

Using expressions for stresses σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{zx} (given by expressions (8.1)–(8.5), respectively) and definitions (given by Eq. (4.8)), one gets:

$$N_x = \overbrace{\frac{Eh}{1-\mu^2}\left(\frac{\partial u_o}{\partial x} + \mu \frac{\partial v_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o}, \tag{9.1}$$

$$N_y = \overbrace{\frac{Eh}{1-\mu^2}\left(\frac{\partial v_o}{\partial y} + \mu \frac{\partial u_o}{\partial x}\right)}^{\text{contribution due to } u_o \text{ and } v_o}, \tag{9.2}$$

$$N_{xy} = \overbrace{\frac{Eh}{2(1+\mu)}\left(\frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o}, \tag{9.3}$$

$$M_x = \overbrace{-D\left(\frac{\partial^2 w_b}{\partial x^2} + \mu \frac{\partial^2 w_b}{\partial y^2}\right)}^{\text{contribution due to } u_b, v_b}, \tag{9.4}$$

$$M_y = \overbrace{-D\left(\frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2}\right)}^{\text{contribution due to } u_b, v_b}, \tag{9.5}$$

$$M_{xy} = \overbrace{-D\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right]}^{\text{contribution due to } u_b, v_b}, \tag{9.6}$$

$$Q_x = \overbrace{-D\frac{\partial}{\partial x}(\nabla^2 w_b)}^{\text{contribution due to } u_{c_1}, w_{c_1}} + \overbrace{\frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right)}^{\text{contribution due to } u_i, w_i}, \tag{9.7}$$

$$Q_y = \overbrace{-D\frac{\partial}{\partial y}(\nabla^2 w_b)}^{\text{contribution due to } v_{c_1}, w_{c_1}} + \overbrace{\frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial y}\right)}^{\text{contribution due to } v_i, w_i}, \tag{9.8}$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)}. \tag{9.9}$$

10. Displacement expressions for particular classes of problems

Displacements u, v, w are given by Eqs. (7.1)–(7.3), respectively. However, for specific cases, all the terms mentioned in the equations would not be required. The terms required for specific cases would now be stated:

1) *Displacements in the case of plane stress*

   Displacements u, v, w in the case of plane stress would only include terms which represent contributions due to components u_o, v_o, u_{o_c}, v_{o_c}, w_{o_c} from amongst the terms mentioned on the right hand side of Eqs. (7.1)–(7.3), respectively.
2) *Displacements in the case of plate flexure*

   Displacements u, v, w in the case of flexure would only include terms which represent contributions due to components u_b, v_b, w_b, u_{c_1}, v_{c_1}, w_{c_1}, u_{c_2}, v_{c_2}, w_{c_2} from amongst the terms mentioned on the right hand side of Eqs. (7.1)–(7.3), respectively.
3) *Displacements in the case of free vibrations*

   Displacements u, v, w in the case of free vibrations would only include terms which represent contributions due to components u_b, v_b, w_b, u_{c_1}, v_{c_1}, w_{c_1}, u_i, v_i, w_i from amongst the terms mentioned on the right hand side of Eqs. (7.1)–(7.3), respectively.

11. Expressions for stresses σ_x, σ_y, τ_{xy} and τ_{yz}, τ_{zx} for particular classes of problems (using displacements, strain definitions, constitutive relations)

In Section 8, general expressions for stresses σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{zx} (given by Eqs. (8.1)–(8.5), respectively) were obtained. However, for specific cases, all the terms mentioned in these equations would not be required. The terms required for specific cases would now be stated.

1) *In the case of plane stress*

   (a) Expressions for stresses σ_x, σ_y, τ_{xy} in the case of plane stress (using displacements, strain definitions, constitutive relations)
   Stresses σ_x, σ_y, τ_{xy} in the case of plane stress would only include terms which represent contributions due to components u_o, v_o, u_{o_c}, v_{o_c} from amongst the terms mentioned in the right hand sides of Eqs. (8.1)–(8.3), respectively.

   (b) Expressions for stresses τ_{yz}, τ_{zx} in the case of plane stress (using displacements, strain definitions, constitutive relations)

It can be seen that u_o, v_o (*plane stress* components of displacements u, v, respectively), and u_{o_c}, v_{o_c}, w_{o_c} (*complimentary plane stress* components of displacements u, v, w, respectively) do not contribute towards tranverse shear stresses, and this gets reflected in the right hand side of Eqs. (8.4) and (8.5), therefore:

$$\tau_{yz} = 0 \quad \text{in the case of plane stress,} \tag{11.1}$$

$$\tau_{zx} = 0 \quad \text{in the case of plane stress.} \tag{11.2}$$

2) *In the case of plate flexure*

(a) Expressions for stresses σ_x, σ_y, τ_{xy} in the case of plate flexure (using displacements, strain definitions, constitutive relations)

The plate flexure is a static phenomenon and involves only lateral loading. There is no in-plane loading.

Therefore, stresses σ_x, σ_y, τ_{xy} in the case of plate flexure would only include terms which represent contributions due to components u_b, v_b, u_{c_1}, v_{c_1}, u_{c_2}, v_{c_2} from amongst the terms mentioned on the right hand sides of Eqs. (8.1)–(8.3), respectively.

(b) Expressions for stresses τ_{zx}, τ_{yz} in the case of plate flexure (using displacements, strain definitions, constitutive relations)

The plate flexure is a static phenomenon and involves only lateral loading. There is no in-plane loading.

Stresses τ_{yz}, τ_{zx} in the case of plate flexure would only include terms which represent contributions due to components u_{c_1}, v_{c_1}, w_{c_1} from amongst the terms mentioned on the right hand sides of Eq. (8.4), (8.5), respectively. Therefore:

$$\tau_{yz} = \overbrace{\frac{Eh^2}{1-\mu^2}\left[\frac{1}{2}\left(\frac{z}{h}\right)^2 - \frac{1}{8}\right]\frac{\partial}{\partial y}(\nabla^2 w_b)}^{\text{contribution due to } v_{c_1} \text{ and } w_{c_1}} \quad \text{in the case of flexure,} \tag{11.3}$$

$$\tau_{zx} = \overbrace{\frac{Eh^2}{1-\mu^2}\left[\frac{1}{2}\left(\frac{z}{h}\right)^2 - \frac{1}{8}\right]\frac{\partial}{\partial x}(\nabla^2 w_b)}^{\text{contribution due to } u_{c_1} \text{ and } w_{c_1}} \quad \text{in the case of flexure.} \tag{11.4}$$

3) *In the case of plate vibrations*

(a) Expressions for stresses σ_x, σ_y, τ_{xy} in the case of plate vibrations (using displacements, strain definitions, constitutive relations)

In free vibrations of plates, there is no lateral loading and there is no in-plane loading.

Stresses σ_x, σ_y, τ_{xy} in the case of plate flexure would only include terms which represent contributions due to components u_b, v_b, u_{c_1}, v_{c_1}, u_i, v_i from amongst the terms mentioned in the right hand sides of Eqs. (8.1)–(8.3), respectively.

(b) Expressions for stresses τ_{zx}, τ_{yz} for vibrations (using displacements, strain definitions, constitutive relations)

In free vibrations of plates, there is no lateral loading and there is no in-plane loading. Stresses τ_{yz}, τ_{zx} for vibrations would be given by Eqs. (8.4), (8.5), respectively.

12. Approximate, yet accurate and easy to use expressions for stresses σ_x, σ_y, τ_{xy} and σ_z in the case of plane stress

Stresses σ_x, σ_y, τ_{xy} in the case of plane stress can be obtained as shown in item 1 of Section 11. However, it is possible to arrive at approximate, yet accurate and easy to use expressions for the stresses σ_x, σ_y, τ_{xy} in the case of plane stress.

1) *Approximate, yet accurate and easy to use expressions for σ_x, σ_y, τ_{xy} in the case of plane stress.* It can be observed that:

(a) In the in-plane forces N_x, N_y, N_{xy} (given by expressions (9.1)–(9.3), respectively), there is contribution only from u_o, v_o.

(b) In the general expressions for stresses σ_x, σ_y, τ_{xy} (given by expressions (8.1)–(8.3), respectively), there are contributions from u_o, v_o and from u_{o_c}, v_{o_c}. And, it can be observed that:

i) The contributions towards stresses σ_x, σ_y, τ_{xy} due to u_o, v_o are constant across the thickness and only these terms contribute towards forces N_x, N_y, N_{xy}.

ii) The contributions towards σ_x, σ_y, τ_{xy} due to u_{o_c}, v_{o_c} involve a function of z containing the higher order h^2 term (and, therefore of lesser importance and insignificant). *And, not withstanding that, and most importantly, u_{o_c}, v_{o_c} do not at all contribute towards in-plane forces N_x, N_y, N_{xy}.*

As a result, in the general expressions for σ_x, σ_y, τ_{xy} (given by expressions (8.1)–(8.3)), the contributions of complimentary plane stress components u_{o_c}, v_{o_c} can be safely ignored.

Therefore, in the case of plane stress, the stresses σ_x, σ_y, τ_{xy} can be given by approximate, yet accurate and easy to use expressions:

(12.1) $$\sigma_x \approx \overbrace{\frac{E}{1-\mu^2}\left(\frac{\partial u_o}{\partial x} + \mu\frac{\partial v_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o} \quad \text{in the case of plane stress,}$$

(12.2) $$\sigma_y \approx \overbrace{\frac{E}{1-\mu^2}\left(\frac{\partial v_o}{\partial y} + \mu\frac{\partial u_o}{\partial x}\right)}^{\text{contribution due to } u_o \text{ and } v_o} \quad \text{in the case of plane stress,}$$

(12.3) $$\tau_{xy} \approx \overbrace{\frac{E}{2(1+\mu)}\left(\frac{\partial v_o}{\partial x} + \frac{\partial u_o}{\partial y}\right)}^{\text{contribution due to } u_o \text{ and } v_o} \quad \text{in the case of plane stress.}$$

2) *In-plane force expressions N_x, N_y, N_{xy} do not change even though they are obtained using approximate in-plane stresses σ_x, σ_y, τ_{xy}.* For obtaining in-plane stresses σ_x, σ_y, τ_{xy} there would be two options:

   (a) In-plane stresses σ_x, σ_y, τ_{xy} can be obtained by using Eqs. (12.1)–(12.3), respectively, or

   (b) for marginally higher accuracy, in-plane stresses σ_x, σ_y, τ_{xy} can be obtained as specified in item 1(a) of Section 11.

   However, whether expressions for in-plane stresses σ_x, σ_y, τ_{xy} are obtained by either using just the mentioned option (a) or option (b), in both the cases N_x, N_y, N_{xy} are still given by expressions (9.1)–(9.3), respectively.

3) *Expression for σ_z and satisfaction of constitutive relation* (4.4) *in the case of plane stress*: As per assumption 2 of Section 5, in the case of the plane stress problem of the plate, as there is no lateral loading, and as transverse strains are not prevented, transverse normal stress σ_z would hardly be there; and it can be safely assumed that: $\sigma_z = 0$. Therefore:

(12.4) $$\sigma_z = 0 \quad \text{in the case of plane stress.}$$

   The displacemens associated with the plane stress case are specified in item 1 of Section 10. In the displacement w, given by Eq. (7.3), only the w_{o_c} (*complimentary plane stress* component of displacement w) is associated with the plane stress case. Therefore, one gets:

(12.5) $$\epsilon_z = -\frac{\mu}{1-\mu}\left(\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y}\right) \quad \text{in the case of plane stress.}$$

   Using Eqs. (12.1), (12.2), (12.4), (12.5) in the constitutive relation (4.4), one can see that the constitutive relation (4.4) gets satisfied in the case of plane stress.

13. Approximate, yet accurate and easy to use expressions for stresses σ_x, σ_y, τ_{xy} and σ_z in the case of flexure, vibrations

In the case of plate flexure, stresses σ_x, σ_y, τ_{xy} can be obtained as specified in item 2(a) of Section 11.

In the case of vibrations, stresses σ_x, σ_y, τ_{xy} can be obtained as specified in item 3(a) of Section 11.

However, it is possible to arrive at approximate, yet accurate and easy to use expressions for the stresses σ_x, σ_y, τ_{xy} in cases of flexure and vibrations.

1) *Approximate, yet accurate and easy to use expressions for stresses σ_x, σ_y, τ_{xy} in cases of flexure and vibrations.* It can be observed that:
   (a) In moments M_x, M_y, M_{xy} (given by expressions (9.4)–(9.6), respectively) there are contributions only from u_b, v_b.
   (b) In the general expressions of stresses σ_x, σ_y, τ_{xy} (given by expressions (8.1), (8.2), (8.3), respectively) there are contributions from u_b, v_b, and also from u_{c_1}, v_{c_1}, u_{c_2}, v_{c_2}, u_i, v_i. It can be seen that:
      i) The contributions towards stresses σ_x, σ_y, τ_{xy} from the displacement components u_b, v_b contain linear z terms, and only these terms contribute towards moments M_x, M_y, M_{xy}.
      ii) The contributions towards stresses σ_x, σ_y, τ_{xy} from the complimentary displacement components u_{c_1}, v_{c_1}, u_{c_2}, v_{c_2}, u_i, v_i also involve functions of z. And, these functions in case of u_{c_1}, v_{c_1} contain h^3 terms; in case of u_{c_2}, v_{c_2} contain h^2 terms; in case of u_i, v_i contain h^3 terms; and all these are higher order terms (and, therefore, are of less importance). *And, not withstanding that, and most importantly, these terms do not at all contribute towards the moments M_x, M_y, M_{xy}.*

   As a result, in the general expressions for stresses σ_x, σ_y, τ_{xy} (given by expressions (8.1)–(8.3), respectively), the contribution of displacement components u_{c_1}, v_{c_1}, u_{c_2}, v_{c_2}, u_i, v_i can be safely ignored.

Therefore, in the case of plate flexure and in the case of plate vibrations, one can write approximate, yet accurate and easy to use expressions for stresses σ_x, σ_y, τ_{xy} as:

$$\sigma_x \approx \overbrace{-\frac{Ez}{1-\mu^2}\left[\frac{\partial^2 w_b}{\partial x^2}+\mu\frac{\partial^2 w_b}{\partial y^2}\right]}^{\text{contribution due to } u_b, v_b} \quad \text{in the cases of flexure and vibrations,} \tag{13.1}$$

$$\sigma_y \approx \overbrace{-\frac{Ez}{1-\mu^2}\left[\frac{\partial^2 w_b}{\partial y^2}+\mu\frac{\partial^2 w_b}{\partial x^2}\right]}^{\text{contribution due to } u_b, v_b} \quad \text{in the cases of flexure and vibrations,} \tag{13.2}$$

$$\tau_{xy} \approx \overbrace{-\frac{Ez}{1-\mu^2}\left[(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y}\right]}^{\text{contribution due to } u_b, v_b} \quad \text{in the case of flexure and vibrations.} \tag{13.3}$$

2) *Moment expressions M_x, M_y, M_{xy} do not change even though they are obtained using approximate stresses σ_x, σ_y, τ_{xy}.* For expressions for stresses σ_x, σ_y, τ_{xy} there would be two options:

   (a) stresses σ_x, σ_y, τ_{xy} can be obtained by using Eqs. (13.1)–(13.3), respectively, or
   (b) for marginally higher accuracy, stresses σ_x, σ_y, τ_{xy} can be obtained for flexure as specified in item 2(a) of Section 11, and can be obtained for vibrations as specified in item 3(a) of Section 11.

However, whether expressions for stresses σ_x, σ_y, τ_{xy} are obtained by either using the just mentioned option (a) or (b), in both the cases M_x, M_y, M_{xy} are still given by expressions (9.4)–(9.6), respectively.

3) *Approximate expression for σ_z and satisfaction of constitutive relation* (4.4) *in the cases of flexure and vibrations.*

   (a) *In the case of plate flexure*: in item 2 of Section 10, displacements u, v, w for the case of flexure are specified. Using the information for w, the following expression would be valid in the case of flexure:

$$\epsilon_z = \frac{\mu z}{1-\mu}(\nabla^2 w_b) + \left[-2\left(\frac{z}{h}\right)^3 + \frac{3}{2}\left(\frac{z}{h}\right) - \frac{1}{2}\right]\frac{q}{E} \quad \text{in the case of flexure.} \tag{13.4}$$

It may be pertinent to note that the constitutive relation (4.4), which involves strain ϵ_z and stresses σ_x, σ_y, σ_z, is completely ignored in many plate theories (e.g., CPT [1], Mindlin's theory [3], RPT [10]).

It has just been shown that expressions (13.1), (13.2), though approximate, are nearly equivalent of expressions (8.1), (8.2), respectively (a numerical example, which would be taken up subsequently in this paper, would substantiate this statement).

Instead of completely ignoring constitutive relation (4.4), it would be better to satisfy the constitutive relation in an approximate, yet practically accurate, manner.

Therefore, using expression (13.1) for σ_x and using expression (13.2) for σ_y in constitutive relation (4.4), one can get:

$$\epsilon_z = \frac{\mu z}{1-\mu}(\nabla^2 w_b) + \frac{\sigma_z}{E} \quad \text{in the case of flexure.} \tag{13.5}$$

Comparing Eq. (13.5) with Eq. (13.4), one can conclude that:

(13.6) $$\sigma_z = \left[-2\left(\frac{z}{h}\right)^3 + \frac{3}{2}\left(\frac{z}{h}\right) - \frac{1}{2}\right]q \quad \text{in the case of flexure.}$$

In item 2(b) of Section 3, loading on the plate in case of flexure was described. The stress σ_z, given by the expression (13.6), satisfies the surface conditions:

(13.7) $$[\sigma_z]_{z=h/2} = 0,$$

(13.8) $$[\sigma_z]_{z=-h/2} = -q.$$

(b) *In the case of plate vibrations*: displacements u, v, w for the case of vibrations are specified in item 3 of Section 10. Using this information in the case of w, the following expression would be valid in the case of vibrations:

(13.9) $$\epsilon_z = \frac{\mu z}{1-\mu}(\nabla^2 w_b) \quad \text{in the case of vibrations.}$$

As per assumption 2 of Section 5, in the case of plate vibrations, as there is no lateral loading, and as transverse strains are not prevented, transverse normal stress σ_z would hardly be there; and it can be safely assumed that: $\sigma_z = 0$. Therefore:

(13.10) $$\sigma_z = 0 \quad \text{in the case of vibrations.}$$

Using Eqs. (13.9), (13.1), (13.2), (13.10) in the constitutive relation (4.4), one can see that the constitutive relation (4.4) gets satisfied.

14. Obtaining of expressions for axial forces, moments and shear forces in particular cases

Expressions for axial forces, moments, shear forces were obtained in Section 9. These expressions were obtained by using displacements, strain definitions, constitutive relations, definitions for axial forces, moments, shear forces.

In this context, the contents of item 2 of Section 12, and the contents of item 2 of Section 13 need to be noted.

Expressions for particular cases would now be mentioned.

1) *Forces N_x, N_y, N_{xy} in the case of plane stress.* In the plane stress case, in-plane forces N_x, N_y, N_{xy} are given by Eqs. (9.1)–(9.3), respectively.
2) *Moments M_x, M_y, M_{xy} in the case of plate flexure and plate vibrations.* In the plate flexure case and also in the plate vibration case, moments M_x, M_y, M_{xy} are given by Eqs. (9.4)–(9.6), respectively.

3) *Shear forces Q_x, Q_y in the case of plate flexure.* General expressions for Q_x, Q_y were given by Eqs. (9.7), (9.8). In the case of plate flexure, there would not be time dependent terms. Therefore, in the case of plate flexure, shear forces Q_x, Q_y are as follows:

$$Q_x = \overbrace{-D\,\frac{\partial}{\partial x}(\nabla^2 w_b)}^{\text{contribution due to } u_{c_1},\, w_{c_1}} \quad \text{in the case of flexure,} \tag{14.1}$$

$$Q_y = \overbrace{-D\frac{\partial}{\partial y}(\nabla^2 w_b)}^{\text{contribution due to } v_{c_1},\, w_{c_1}} \quad \text{in the case of flexure.} \tag{14.2}$$

4) *Shear forces Q_x, Q_y in the case of plate vibrations.* In the plate vibration case, shear forces Q_x, Q_y are given by Eqs. (9.7) and (9.8), respectively.

15. Summary of expressions of various entities of the Improved 3D-RPT

1) Plane stress case:

   (a) u, v, w as mentioned in item 1 of Section 10.
   (b) σ_x, σ_y, τ_{xy} are as mentioned in item 1(a) of Section 11
   or
   are as given by *approximate, yet accurate and easy to use* Eqs. (12.1)–(12.3), respectively.
   (c) σ_z, τ_{yz}, τ_{zx} are given by Eqs. (12.4), (11.1), (11.2), respectively.
   (d) N_x, N_y, N_{xy} are given by Eqs. (9.1)–(9.3), respectively.

2) Plate flexure case:

   (a) u, v, w as mentioned in item 2 of Section 10.
   (b) σ_x, σ_y, τ_{xy} are as mentioned in item 2(a) of Section 11 *or* are given by *approximate, yet accurate and easy to use* Eqs. (13.1)–(13.3), respectively.
   (c) σ_z, τ_{yz}, τ_{zx} are given by Eqs. (13.6), (11.3), (11.4), respectively.
   (d) M_x, M_y, M_{xy} are given by Eqs. (9.4)–(9.6), respectively.
   (e) Q_x, Q_y are given by Eqs. (14.1), (14.2), respectively.

3) Free vibration case:

   (a) u, v, w as mentioned in item 3 of Section 10.
   (b) σ_x, σ_y, τ_{xy} are as mentioned in item 3(a) of Section 11
   or

are given by *approximate, yet accurate and easy to use* Eqs. (13.1)–(13.3), respectively.

(c) σ_z, τ_{yz}, τ_{zx} are given by Eqs. (13.10), (8.4), (8.5), respectively.

(d) M_x, M_y, M_{xy} are given by Eqs. (9.4)–(9.6), respectively.

(e) Q_x, Q_y are given by Eqs. (9.7), (9.8), respectively.

4) Plate stability case:

In Section 5, as per assumptions 6(a) and 6(b), it can be assumed that the case of static stability of plate can be considered to be a combination of the case of the plane stress problem (involving in-plane loading for evaluation of in-plane forces) and a case of the plate flexure problem (involving lateral loading for evaluation of moments, shear forces).

Therefore, contents of just mentioned items 1 and 2 (of the current Section 15) are also applicable for the case of plate stability.

16. General gross equilibrium equations in Improved 3D-RPT

The use of the variational approach would have resulted in higher-order terms in boundary conditions, governing equations. In [10], it was shown that both the gross equilibrium and the variational approaches give almost identical results. Therefore, to keep the theory simple and also accurate, the gross equilibrium approach would be used in this paper.

The theory of elasticity equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \rho \frac{\partial^2 u}{\partial t^2} = 0, \tag{16.1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} - \rho \frac{\partial^2 v}{\partial t^2} = 0, \tag{16.2}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} - \rho \frac{\partial^2 w}{\partial t^2} = 0. \tag{16.3}$$

Integrating Eqs. (16.1) and (16.2) with respect to z and between the integration limits $z = -h/2$ and $z = h/2$, using definitions (4.8), noting that plate surfaces are shear stress free, and as per item 2(a) of Section 3 in-plane forces are assumed to be static in nature, one gets:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{16.4}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \tag{16.5}$$

Multiplying Eq. (16.1) by z, similarly multiplying Eq. (16.2) by z, and then integrating with respect to z and between the integration limits of $z = -h/2$ and $z = h/2$, and noting definitions (4.8) and Eqs. (7.1), (7.2), (8.6) one gets:

$$\text{(16.6)} \qquad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right) = 0,$$

$$\text{(16.7)} \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial y}\right) = 0.$$

Using expressions (9.4) through (9.8), it can be seen that equilibrium equations (16.6), (16.7) get satisfied identically.

Integrating Eq. (16.3) with respect to z and between integration limits of $z = -h/2$ and $z = h/2$, and noting definitions (4.8) and Eqs. (7.3), (13.7), (13.8) one gets:

$$\text{(16.8)} \qquad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - \rho \int\limits_{z=-h/2}^{z=h/2} \frac{\partial^2 w}{\partial t^2} = 0.$$

Using expression (7.3) in Eq. (16.8), and noting (as per item 2(b) of Section 3) lateral loading q is static in nature and not a function of time, one gets:

$$\text{(16.9)} \qquad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - \rho h\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho h^3}{12}\,\frac{12-\mu}{5(1-\mu)}\,\frac{\partial^2}{\partial t^2}(\nabla^2 w_b) - \frac{(1+\mu)}{5}\,\frac{\rho^2 h^3}{E}\,\frac{\partial^4 w_b}{\partial t^4} = 0.$$

Equation (16.9) can be expressed in terms of moments. By using Eqs. (16.6), (16.7) one can obtain expressions for shear forces Q_x, Q_y in terms of moments M_x, M_{xy}, M_y; and using this information in Eq. (16.9), one obtains:

$$\text{(16.10)} \qquad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q - \rho h\frac{\partial^2 w_b}{\partial t^2} + \frac{\rho h^3}{12}\left[1 + \frac{12-\mu}{5(1-\mu)}\right]\frac{\partial^2}{\partial t^2}(\nabla^2 w_b) - \frac{(1+\mu)}{5}\,\frac{\rho^2 h^3}{E}\,\frac{\partial^4 w_b}{\partial t^4} = 0.$$

Equations (16.4), (16.5), (16.10) are the gross equlibrium equations of the Improved 3D-RPT in terms of axial forces, moments. These gross equilibrium equations would be utilized to yield governing equations for specific cases of plate flexure, vibrations, stability.

17. Governing equations for specific cases in the Improved 3D-RPT

Governing equations for flexure, vibrations and for the plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT would now be stated.

17.1. Governing equation for the plane stress in the Improved 3D-RPT

The plane stress case involves only N_x, N_y, N_{xy}. Therefore, Eqs. (16.4), (16.5) would be the governing equations for the plane stress case in the Improved 3D-RPT. These equations are again quoted below for the sake of convenience:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{17.1}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \tag{17.2}$$

In view of assumption 6(a) of Section 5, Eqs. (17.1), (17.2) would remain valid even when the in-plane forces and lateral loading both are applied.

17.2. Governing equation for the plate flexure in the Improved 3D-RPT

For the case of plate flexure, as per item 2(b) of Section 3, it was assumed that the plate carries only a static lateral load of an intensity $q(x, y)$ and there is no in-plane loading. Using expressions for M_x, M_y, M_{xy} (given by expressions (9.4)–(9.6), respectively) in Eq. (16.10), and noting that forces N_x, N_y, N_{xy}, and time dependant terms would not be there, one gets:

$$\bigtriangledown^2 \bigtriangledown^2 w_b = \frac{q}{D}. \tag{17.3}$$

Equation (17.3) is the governing equation for the plate flexure in the Improved 3D-RPT.

Remark. The governing equation (17.3) of the Improved 3D-RPT has striking similarity with that of the CPT save for the appearance of w_b in Eq. (17.3) of the Improved 3D-RPT, whearas w appears in the context of the CPT.

17.3. Governing equation for the free vibrations in the Improved 3D-RPT

For free vibrations, as per item 2(c) of Section 3, it was assumed that lateral loading q and in-plane forces (N_x, N_y, N_{xy}) are absent, then using expressions for M_x, M_y, M_{xy} (given by expressions (9.4)–(9.6), respectively) in Eq. (16.10), one gets:

$$D \nabla^2 \nabla^2 w_b + \rho h \frac{\partial^2 w_b}{\partial t^2} - \frac{\rho h^3}{12}\left[1 + \frac{12-\mu}{5(1-\mu)}\right]\frac{\partial^2}{\partial t^2}(\nabla^2 w_b) + \frac{(1+\mu)}{5}\frac{\rho^2 h^3}{E}\frac{\partial^4 w_b}{\partial t^4} = 0. \tag{17.4}$$

Equation (17.4) is the governing equation for the free vibrations in the Improved 3D-RPT.

17.4. Governing equation for the plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT

As per item 2(d) of Section 3, the loading would be a combination of the cases of plane stress and flexure (i.e., a combination of cases mentioned in items 2(a) and 2(b) of Section 3). The lateral load and in-plane forces are static in nature, and the in-plane forces are applied at the mid-surface of the plate.

As per items 6(a) and 6(b) of Section 5, it was assumed that in-plane stresses due to in-plane loads remain practically the same whether lateral load is applied or not. Also, moments and shear forces in the plate due to applied lateral loading remain practically the same whether additional in-plane forces are applied or not.

It may be noted that the just mentioned assumptions are in tune with the discussion in respect of the thin plate (subjected to combined action of in-plane forces and lateral forces) in [1, pp. 378 and 385].

Following exactly the same logic (as given in [1, pp. 378–379], which was used for formulation of the problem of a thin plate under combined action of lateral loading and in-plane forces), in-plane forces N_x, N_y, N_{xy} would be obtained, then their effect on equilibrium in the z-direction would be considered. The following steps are involved:

1) Equations (17.1), (17.2) are the equilibrium equations in the x-direction and in the y-direction, respectively.

   There are two possibilities about in-plane forces N_x, N_y, N_{xy}:

   (a) If in-plane forces N_x, N_y, N_{xy} are specified such that they are in equilibrium by themselves, then the equilibrium equations (17.1), (17.2) would get automatically satisfied.

   (b) Otherwise, the equilibrium equations (17.1), (17.2) can be solved as a plane stress problem taking care of plane stress boundary conditions of the problem, and this may involve, depending on the problem, u_o, v_o as unknowns to be found out.

   In general, plane stress problems are comparatively easy to solve. After solving equations (17.1), (17.2) in-plane forces N_x, N_y, N_{xy} can be obtained.

2) Using Eq. (16.10), and noting that the loading involved in the present case is static in nature, the equilibrium equation in the z-direction in terms of displacements would now be obtained.

Notation $w_o(x, y)$ would be used to denote mid-surface displacement $[w(x, y)]_{z=0}$. Using Eq. (7.3), one can write:

$$w_o(x, y) = [w(x, y)]_{z=0} = w_b(x, y) - h^2 \frac{(8+\mu)}{40(1-\mu)} (\nabla^2 w_b) - h \frac{39}{1120} \frac{q}{E}. \tag{17.5}$$

From Eq. (17.5), it may be noted that w_o involves only w_b as an unknown function (as the applied lateral load $q(x, y)$ is a known entity).

While considering the equilibrium in the z-direction, the components of forces N_x, N_y, N_{xy} in the z-direction need to be taken into account. The resultant equilibrium equation can then be written as:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left[q + N_x \frac{\partial^2 w_o}{\partial x^2} + 2N_{xy} \frac{\partial^2 w_o}{\partial x \partial y} + N_y \frac{\partial^2 w_o}{\partial y^2}\right]. \tag{17.6}$$

Using expressions for M_x, M_y, M_{xy} (given by expressions (9.4)–(9.6), respectively) in Eq. (17.6) one gets:

$$\frac{\partial^4 w_b}{\partial x^4} + 2\frac{\partial^4 w_b}{\partial x^2 \partial y 2} + \frac{\partial^4 w_b}{\partial y^4} = \frac{1}{D}\left[q + N_x \frac{\partial^2 w_o}{\partial x^2} + 2N_{xy} \frac{\partial^2 w_o}{\partial x \partial y} + N_y \frac{\partial^2 w_o}{\partial y^2}\right]. \tag{17.7}$$

Equation (17.7) is the equilibrium equation in the z-direction. It should be noted that the entity w_o (given by Eq. (17.5)) appearing in Eq. (17.7) involves only w_b as an unknown variable.

Equations (17.1), (17.2), (17.7) are the governing equations for the plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT.

18. Boundary conditions in the Improved 3D-RPT

For illustration purpose, boundary conditions would be discussed on the edge $x = a$ of a plate having geometry described by Eq. (3.1). The boundary conditions would be prescribed by physical understanding of the problem.

18.1. Boundary conditions for plane stress in the Improved 3D-RPT

Plane stress boundary conditions on the edge $x = a$:

(18.1) either $[N_x]_{x=a} = 0$ or $[u_o]_{x=a}$ is prescribed,

(18.2) either $[N_{xy}]_{x=a} = 0$ or $[v_o]_{x=a}$ is prescribed.

18.2. Boundary conditions for plate flexure in the Improved 3D-RPT

1) If the edge $x = a$ is simply supported, then:

$$[w]_{x=a,\,z=0} = 0 \quad \text{and} \quad [M_x]_{x=a} = 0. \tag{18.3}$$

2) If the edge $x = a$ is clamped:
   Unlike the CPT, two types of boundary conditions analogous to those discussed (in the context of the theory of elasticity solutions for beams) by Timoshenko and Goodier [23, pp. 45–46] are possible (similar was also the case in RPT-Variant II [10]).
   Groh and Weaver [24] have observed static inconsistencies that arise when modelling the flexural behaviour of beams, plates and shells with clamped boundary conditions using a certain class of axiomatic, higher-order shear deformation theories. The clamped boundary conditions going to be prescribed here do not suffer from such inconsistencies.

   (a) If the edge $x = a$ is clamped type I, then:

$$[w]_{x=a,\,z=0} = 0 \quad \text{and} \quad \left[\frac{\partial w}{\partial x}\right]_{x=a,\,z=0} = 0. \tag{18.4}$$

   (b) If the edge $x = a$ is clamped type II, then:

$$[w]_{x=a,\,z=0} = 0 \quad \text{and} \quad \left[\frac{\partial u}{\partial z}\right]_{x=a,\,z=0} = 0. \tag{18.5}$$

3) If the edge $x = a$ is free, then:

$$[M_x]_{x=a} = 0 \quad \text{and} \quad \left[Q_x + \frac{\partial M_{xy}}{\partial y}\right]_{x=a} = 0, \tag{18.6}$$

   where Q_x is given by Eq. (14.1) in the case of flexure.

18.3. Boundary conditions for plate vibration in the Improved 3D-RPT

The boundary conditions are the same as those given in the case of plate flexure (as mentioned in Section 18.2), *but with the following exception: In the case of plate vibration, in Eq.* (18.6), Q_x *is given by Eq.* (9.7).

18.4. Boundary conditions for plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT

1) Plane stress boundary conditions on the edge $x = 0$ are prescribed by Eqs. (18.1) and (18.2).
2) (a) If the edge $x = a$ is simply supported, then conditions are prescribed by Eq. (18.3).

(b) i) If the edge $x = a$ is clamped of type I, then conditions are prescribed by Eq. (18.4).

ii) If the edge $x = a$ is clamped of type II, then conditions are prescribed by Eq. (18.5).

(c) If the edge $x = a$ is free, then conditions are prescribed by (18.6).

19. Some noteworthy significant similarities of the Improved 3D-RPT with the CPT and other theories

1) Some of the expressions for stresses, moments, forces of the Improved 3D-RPT are as follows:

(a) *Plate flexure*:

Stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}	given by Eqs. (13.1), (13.2), (13.6), (13.3), (11.3), (11.4), respectively;
Moments M_x, M_y, M_{xy}	given by Eqs. (9.4)–(9.6), respectively;
Shear Forces Q_x, Q_y	given by Eqs. (14.1), (14.2), respectively.

(b) *Plate vibrations*:

Stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}	given by Eqs. (13.1), (13.2), (13.10), (13.3), (8.4), (8.5), respectively;
Moments M_x, M_y, M_{xy}	given by Eqs. (9.4)–(9.6), respectively;
Shear forces Q_x, Q_y	given by Eqs. (9.7), (9.8), respectively.

(c) *Plane stress*:

Stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}	given by Eqs. (12.1), (12.2), (12.4), (12.3), (11.1), (11.2), respectively;
In-plane forces N_x, N_y, N_{xy}	given by Eqs. (9.1)–(9.3), respectively.

The equations of the Improved 3D-RPT, mentioned against (a) *Plate flexure* and (b) *Plate vibrations* have striking similarities with the corresponding equations of the CPT save for the appearance of w_b in the case of the Improved 3D-RPT, whearas w appears in the context of the CPT.

The equations of the Improved 3D-RPT, mentioned against (c) *Plane stress* have striking similarities with the corresponding equations of the conventional two-dimensional theory of elasticity save for the appearance of u_o, v_o in the case of the Improved 3D-RPT, whearas usually u, v appear in the context of the two-dimensional theory of elasticity.

2) Similarities about governing equations, boundary conditions of the Improved 3D-RPT

(a) *Plate flexure*:

The number of unknown variables involved in the governing equation (17.3) of the Improved 3D-RPT is only one (as is the case with the CPT).

The governing equation (17.3) of the Improved 3D-RPT has striking similarity with that of the CPT save for the appearance of w_b in Eq. (17.3) of the Improved 3D-RPT, whearas w appears in the context of the CPT.

Except for the clamped type II boundary conditions (18.5), other boundary conditions of the Improved 3D-RPT have good amount of similarity with those of the CPT.

(b) *Plate vibrations*:

The number of unknown variables involved in the governing equation (17.4) of the Improved 3D-RPT is only one (as is the case with the CPT).

In the governing equation (17.4) for vibration of the Improved 3D-RPT, there is a square bracket $\left[1+\frac{12-\mu}{5(1-\mu)}\right]$. Instead of that square bracket, another square bracket $\left[1+\frac{12}{5(1-\mu)}\right]$ appears in the corresponding Eq. (50) of [20], which is derived using "A single variable refined theory for free vibrations of a plate using inertia related terms in displacements" (SVRPT), and the theory does not take into account normal stress σ_z.

There is a discussion in [20, p. 143] about the governing equation (50) of [20]. In view of that the following can be stated:

There is a significant similarity of the governing equation (17.4) of the Improved 3D-RPT with some equations of the following earlier theories:

i) Mindlin's theory [3],
ii) Levinson's theory [6], and
iii) "A single variable refined theory for free vibrations of a plate using inertia related terms in displacements" (SVRPT) [20].

But, one can note that, MINDLIN [3] and LEVINSON [6] theories involve three unknown functions. The SVRPT [20] involved only one unknown function. All the three theories take into account transverse shear stresses, but do not take into account σ_z. In contrast, the present theory involves only one unknown function and takes into account transverse shear stresses, as well as, the transverse normal stress σ_z.

(c) *Plate under combined action of lateral loads and in-plane forces in the improved 3D-RPT*:

there are three governing equations:

i) Equations (17.1), (17.2) of the Improved 3D-RPT are similar to the two equations given in [1, p. 378].

ii) Equation (17.7) of the Improved 3D-RPT is strikingly similar to the Eq. (217) given in [1, p. 379] – the only difference is the appearance of w_b, w_o in Eq. (17.7) of the Improved 3D-RPT instead of w in Eq. (217) of [1]. It should be noted that the entity w_o (given by Eq. (17.5)) appearing in Eq. (17.7) involves only w_b as an unknown variable.
And, just mentioned Eq. (217) of [1] is the same as that obtained by Saint-Venant, if the body forces are ignored (as per the information in [1, p. 380]).
Therefore, the governing equation (17.7) of the Improved 3D-RPT has also a noteworthy significant similarity to the corresponding equation obtained by Saint-Venant in respect of displacement of the thin plate subjected to in-plane and lateral loading.

For a plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT, the following needs to be noted:

i) In general, as is the case with the thin plate, in the case of the Improved 3D-RPT would involve three unknown variables, and these unknown variables are u_o, v_o, w_b.
Three Eqs. (17.1), (17.2), (17.7) need to be solved.

ii) However, quite often, known in-plane forces are prescribed, which inherently satisfy governing equations (17.1), (17.2), then only one unknown variable is involved.
In that case Eq. (17.7), which involves only one unknown variable w_b, needs to be satisfied.

20. About the procedures to obtain stresses in the CPT, and the Improved 3D-RPT, the exact theories

The difference between procedures to obtain stresses in the CPT and in the Improved 3D-RPT, needs to be noted:

1) In the CPT, only two direct stresses σ_x, σ_y and one shear stress τ_{xy} can be obtained using strain-displacement relations and constitutive relations.
In the CPT, stresses τ_{zx}, τ_{yz}, σ_z can only be obtained by tedious indirect manner by making use of equilibrium equations.

2) Whereas, in the Improved 3D-RPT and also in the exact theories, all the three direct stresses σ_x, σ_y, σ_z and all the three shear stresses τ_{xy}, τ_{yz}, τ_{zx} can be obtained directly using strain-displacement relations, constitutive relations.

21. On the accuracy of the CPT as compared to the exact theories and implications for the Improved 3D-RPT

Exact plate theories were proposed by Srinivas *et al.* [11] and by Pagano [12].

1) It has been observed by Srinivas *et al.* in [11, p. 454]:

> "The most important conclusion to draw from Table 3, is that even for a 14% thick square plate, for which thin plate theory underestimates maximum deflection by 8% and maximum stress by 2%, the true thickness-wise distributions of stresses are very close to the thin plate and Reissner's distributions".

2) As has been observed by Pagano in [12, p. 34]:

> "It appears to be generally true that convergence of the elasticity solution to CPT is more rapid for the stress components than plate deflection, which is an important observation to consider in selecting the form of a plate theory required in the solution of a specific boundary value problem".

In view of the just mentioned observations and noting the contents of item 1 of Section 19 it is a foregone conclusion that in the case of plate flexure the Improved 3D-RPT stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} (given by Eqs. (13.1), (13.2), (13.6), (13.3), (11.3), (11.4), respectively) would be quite accurate. If marginally higher accuracy is desired for stresses σ_x, σ_y, τ_{xy} can be obtained as mentioned in item 2 of Section 11.

However, it needs to be seen whether the displacement w given by Eq. (7.3) of the Improved 3D-RPT is accurate or not. The illustrative example would prove to be useful in that context.

22. Illustrative examples

In the illustrated examples, all sides simply supported a rectangular plate would be considered for flexure, vibrations, stability. The plate occupies the region described by Eq. (3.1). Results are given in Table 1 through Table 4. In the tables, the % errors of particular theories with respect to the exact theory are mentioned in round brackets. These % errors are calculated using the formula:

$$\text{(22.1)} \qquad \%\,\text{error} = \frac{\begin{pmatrix} \text{value by a particular theory} \\ -\text{ corresponding value by the exact theory} \end{pmatrix}}{\text{corresponding value by the exact theory}} \times 100.$$

22.1. Illustrative example – plate flexure

Consider all edges simply supported a rectangular plate, which occupies a region described by Eq. (3.1). The plate carries a uniformly distributed lateral load of intensity q_o (acting in the positive z-direction) on the surface $z = -h/2$. Using standard Navier's technique (described in [1, pp. 108–110]) results are obtained.

The non-dimensional displacement $\widehat{w}$ is defined here as: $\widehat{w} = \frac{Gw}{hq_o}$.

The non-dimensional stress $\widehat{\sigma_x}$ is defined here as: $\widehat{\sigma_x} = \frac{\sigma_x}{q_o}$.

Numerical results for a square plate ($a/b = 1$, $\mu = 0.3$) for various h/a ratios are presented in Tables 1 and 2.

In view of what is stated in Section 21, it is a foregone conclusion that stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} (given by Eqs. (13.1), (13.2), (13.6), (13.3), (11.3), (11.4), respectively) obtained by the Improved 3D-RPT would be quite accurate.

Therefore, results for the displacement w and for the stress σ_x to be obtained as mentioned in item 2(a) of Section 11, for marginally higher accuracy, would need to be observed.

It can be seen from the Table 1 that for a thick plate (having $h/a = 0.14$) the result for $\widehat{w}$ obtained by the CPT has an error of −8.24%. Whereas, the corresponding errors are quite small for other theories: the Reissner (−0.01%), Improved 3D-RPT (−0.15%). Therefore, the Improved 3D-RPT results are fairly accurate.

TABLE 1. Non-dimensional displacement $\widehat{w}$ $\left(= \frac{Gw}{hq_o}\right)$ at ($x = a/2$, $y = a/2$, $z = 0$) for a square plate ($a = b$, $\mu = 0.3$) for various h/a ratios (% error with respect to the exact theory is shown in round brackets).

h/a	CPT [11]	Reissner's theory [11]	Improved 3D-RPT – present theory	Exact theory [11]
0.05	5459.8 (−1.14%)	5519.9 (−0.05%)	5533.2 (0.19%)	5522.5 (0.0%)
0.10	341.24 (−4.39%)	356.27 (− 0.18%)	359.56 (0.75%)	356.9 (0.0%)
0.14	88.827 (−8.24%)	96.497 (−0.01%)	98.159 (−0.15%)	96.801 (0.0%)

From item 2(b) of Section 15, it can be noted that in the Improved 3D-RPT flexural stress σ_x can be obtained using Eq. (13.1), however, for marginally accurate results can be obtained as given in item 2(a) of Section 11. It can be seen from Table 2 that even for a thick plate having $h/a = 0.14$, as was expected in view of the observations made in Section 21, results obtained for $\widehat{\sigma_x}$

TABLE 2. Non-dimensional stress $\widehat{\sigma_x}$ $\left(= \frac{\sigma_x}{q_o}\right)$ at $(x = a/2,\ y = a/2,\ z = -h/2)$ for a square plate ($a = b$, $\mu = 0.3$) for various h/a ratios (% error with respect to the exact theory is shown in round brackets).

h/a	CPT [11]	Reissner's theory [11]	Improved 3D-RPT – present theory using approximate Eq. (13.1)	Improved 3D-RPT – present theory higher accuracy using item 2(a) of Section 11	Exact theory [11]
0.05	−114.93 (−0.29%)	−115.02 (−0.21%)	−114.93 (−0.29%)	−115.09 (−0.15%)	−115.26 (0.0%)
0.10	−28.732 (−0.92%)	−28.822 (−0.61%)	−28.732 (−0.92%)	−28.895 (−0.36%)	−28.998 (0.0%)
0.14	−14.659 (−1.92%)	−14.749 (−1.32%)	−14.659 (−1.92%)	−14.822 (−0.83%)	−14.946 (0.0%)

by all the theories are fairly accurate. Amongst them, the result obtained by Improved 3D-RPT using item 2(a) of Section 11 for higher accuracy has an error of only −0.83%.

22.2. Illustrative example – plate vibration

Consider free vibrations of all edges simply supported a rectangular plate, which occupies a region described by Eq. (3.1). The simply supported boundary conditions would get satisfied, if w_b is assumed as follows:

$$w_b = \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} W_{b_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \omega_{mn} t, \tag{22.2}$$

where $W_{b_{mn}}$ is a constant and ω_{mn} is the circular frequency associated with mode parameters m, n.

Using Eq. (22.2) in governing equation (17.4), one obtains the frequency equation:

$$\left(\frac{\omega_{mn}^2 \rho h^2}{G}\right)^2 - \left(\frac{\omega_{mn}^2 \rho h^2}{G}\right)\left\{10 + \left[\frac{5}{6} + \frac{12-\mu}{6(1-\mu)}\right]\alpha_{mn}\right\} + \left[\frac{5\alpha_{mn}^2}{3(1-\mu)}\right] = 0, \tag{22.3}$$

where

$$\alpha_{mn} = \left(\frac{m\pi h}{a}\right)^2 + \left(\frac{n\pi h}{b}\right)^2.$$

The non-dimensional frequency $\widehat{\omega}_{mn}$ can be defined as: $\widehat{\omega}_{mn} = \omega_{mn} h\sqrt{\rho/G}$.

The results for a thick square plate ($a/b = 1$, $h/a = 0.1$, $\mu = 0.3$) for $\widehat{\omega}_{mn}$ are presented in Table 3.

From Table 3, for a thick square plate (having $h/a = 0.1$), it can be seen that for a higher mode ($m = 4$, $n = 4$) the error in the non-dimensional frequency $\widehat{\omega}_{mn}$ for the CPT is 25.96%. Whereas, the corresponding errors are quite small for other theories: Reddy's theory (−0.39%), Mindlin's theory (−1.15%), RPT (−0.96%), SVRPT (−1.15%), the Improved 3D-RPT (−0.56%). Therefore, the presented theory (Improved 3D-RPT) results are fairly accurate.

TABLE 3. Non-dimensional $\widehat{\omega}_{mn}$ $(= \omega_{mn} h\sqrt{\rho/G})$ predominantely bending frequencies for an isotropic simply supported square plate ($a/b = 1$, $h/a = 0.1$, $\mu = 0.3$) (% errors with respect to exact theory are indicated in round brackets).

m	n	CPT [17]	Reddy's theory [17]	Mindlin's theory [17]	RPT [17]	SVRPT [20]	Improved 3D-RPT – present theory	Exact theory [17]
1	1	0.0955 (2.47%)	0.0931 (−0.11%)	0.0930 (−0.22%)	0.0930 (−0.22%)	0.0930 (−0.22%)	0.0931 (−0.11%)	0.0932 (0.00%)
1	2	0.2360 (6.02%)	0.2222 (−0.18%)	0.2219 (−0.32%)	0.2220 (−0.27%)	0.2219 (−0.32%)	0.2223 (−0.13%)	0.2226 (0.00%)
2	2	0.3732 (9.09%)	0.3411 (−0.29%)	0.3406 (−0.44%)	0.3406 (−0.44%)	0.3406 (−0.44%)	0.3413 (−0.23%)	0.3421 (0.00%)
1	3	0.4629 (10.98%)	0.4158 (−0.31%)	0.4149 (−0.53%)	0.4151 (−0.48%)	0.4149 (−0.53%)	0.4160 (−0.26%)	0.4171 (0.00%)
2	3	0.5951 (13.02%)	0.5221 (−0.34%)	0.5206 (−0.63%)	0.5208 (−0.59%)	0.5206 (−0.63%)	0.5223 (−0.31%)	0.5239 (0.00%)
1	4	0.7668 –	0.6545 –	0.6520 –	0.6525 –	0.6520 –	0.6546 –	– –
3	3	0.8090 (17.43%)	0.6862 (−0.39%)	0.6834 (−0.80%)	0.6840 (−0.71%)	0.6834 (−0.80%)	0.6862 (−0.39%)	0.6889 (0.00%)
2	4	0.8926 (18.84%)	0.7481 (−0.40%)	0.7446 (−0.87%)	0.7454 (−0.76%)	0.7447 (−0.85%)	0.7479 (−0.43%)	0.7511 (0.00%)
3	4	1.0965 –	0.8949 –	0.8896 –	0.8908 –	0.8896 –	0.8942 –	– –
1	5	1.1365 (22.63%)	0.9230 (−0.41%)	0.9174 (−1.01%)	0.9187 (−0.87)	0.9174 (−1.01%)	0.9222 (−0.50%)	0.9268 (0.00%)
2	5	1.2549 –	1.0053 –	0.9984 –	1.0001 –	0.9984 –	1.0040 –	– –
4	4	1.3716 (25.96%)	1.0847 (−0.39%)	1.0764 (−1.15%)	1.0785 (−0.96%)	1.0764 (−1.15%)	1.0828 (−0.56%)	1.0889 (0.00%)

22.3. Illustrative example – stability

Consider all edges of a simply supported rectangular plate, which occupies a region described by Eq. (3.1). The plate carries a lateral load of intensity $q(x, y)$, and, in addition, the plate is also subjected to the in-plane force $N_x = -N_o$. The negative sign associated with N_o indicates the compressive in-plane force.

Since in-plane forces are specified and are in equilibrium, equilibrium equations (17.1), (17.2) are getting satisfied. Therefore, in-plane displacements u_o, v_o would not get involved as unknown variables. And Eq. (17.7) takes the form in this case as:

$$\frac{\partial^4 w_b}{\partial x^4} + 2\frac{\partial^4 w_b}{\partial x^2 \partial y2} + \frac{\partial^4 w_b}{\partial y^4} = \frac{1}{D}\left[q - N_o \frac{\partial^2 w_o}{\partial x^2}\right]. \tag{22.4}$$

To satisfy simply supported boundary conditions on edges, one can express w_b and q as follows:

$$w_b = \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} C_{mn} \sin\frac{m\pi\, x}{a} \sin\frac{n\pi y}{a}, \tag{22.5}$$

$$q = \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,2,\ldots}^{\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{22.6}$$

where C_{mn} and q_{mn} are Fourier constants. Using Eqs. (17.5), (22.5), (22.6) in Eq. (22.4), one gets:

$$C_{mn} = \frac{q_{mn}\left[\dfrac{a^2}{m^2\pi^2} + N_o \dfrac{39}{1120}\dfrac{h}{E}\right]}{\dfrac{\pi^2 D}{b^2}\left(\dfrac{mb}{a} + \dfrac{n^2 a}{mb}\right)^2 - N_o\left[1 + \dfrac{(8+\mu)\,\pi^2}{40(1-\mu)}\left(\dfrac{m^2h^2}{a^2} + \dfrac{n^2h^2}{b^2}\right)\right]}. \tag{22.7}$$

The plate would buckle, when the denominator of Eq. (22.7) becomes zero and, as a result, C_{mn} tends to infinity. In that case the N_{cr} (i.e., the critical value of N_o) when $n = 1$, can be written as:

$$N_{cr} = \frac{\pi^2 D}{b^2} k_{cr}, \tag{22.8}$$

where the non-dimensional parameter k_{cr} is given by:

$$k_{cr} = \frac{\left(\dfrac{mb}{a} + \dfrac{a}{mb}\right)^2}{1 + \dfrac{(8+\mu)\pi^2}{40(1-\mu)}\left(\dfrac{m^2h^2}{a^2} + \dfrac{h^2}{b^2}\right)}. \tag{22.9}$$

As against the non-dimensional k_{cr} given by Eq. (22.9), the corresponding k_{cr} for a thin plate (as given in [1, p. 389]) is $\left(\frac{mb}{a}+\frac{a}{mb}\right)^2$, which incidentally happens to be the numerator of Eq. (22.9).

The values of non-dimensional parameter k_{cr} for a square plate ($a = b$) for various h/b ratios are tabulated in Table 4.

TABLE 4. The values of non-dimensional buckling parameter k_{cr} for a square plate ($a = b$) for various h/b ratios (% error with respect to the exact theory is shown in round brackets).

h/b	CPT [26]	Mindlin's theory [25]	Reddy's theory [25]	Improved 3D-RPT – present theory	Exact theory [26]
0.0	4.000 (0.0 %)	4.000 (0.0 %)	4.000 (0.0 %)	4.000 (0.0 %)	4.000 (0.0 %)
0.05	4.000 (2.28%)	3.944 (0.84%)	3.944 (0.84%)	3.942 (0.79%)	3.911 (0.0%)
0.10	4.000 (6.92%)	3.786 (1.20%)	3.787 (1.23%)	3.779 (1.02%)	3.741 (0.0%)
0.20	4.000 (26.98%)	3.264 (3.62%)	3.265 (3.65%)	3.241 (2.89%)	3.150 (0.0%)

Incidentally, the value of a non-dimensional parameter k_{cr} for a square plate also happens to be the asymptotic value of k_{cr}, when $a \gg b$.

It can be seen from Table 4 that for a thick square plate (having $h/b = 0.2$), if the CPT is used, the value of non-dimensional buckling parameter k_{cr} would have large error of 26.98%. Wheras, the errors are quite small for other theories: Mindlin (3.62%), Reddy (3.65%), the Improved 3D-RPT (2.89%). Amongst them the Improved 3D-RPT has the least error of 2.89%.

22.4. Illustrative example – semi-infinite cantilever plate

In Section 18.2 clamped type I and clamped type II conditions are mentioned. The illustrative example shows the application of the two types of clamped boundary conditions.

Consider a semi-infinite cantilever plate occupying a region:

$$0 \le x \le a, \qquad -\infty \le y \le \infty, \qquad -h/2 \le z \le h/2. \tag{22.10}$$

The semi-infinite cantilever plate dimension in y-direction is infinite. The plate is clamped (the clamping can be either of type I or of type II) at the edge $x = 0$.

There are no constraints on deflection and slope at the edge $x = a$.

The plate does not carry any lateral load on surfaces $z = \pm h/2$.

There are no shear stresses on surfaces $z = \pm h/2$.

A shear load of intensity P per unit length is applied uniformly all along the surface of the edge $x = a$ (i.e., $[V_x]_{x=a} = P$).

Governing equation: Noting the governing equation (Eq. (17.3)) for the plate flexure of the Improved 3D-RPT, and noting that the lateral load q is zero in the present example, the governing equation for the present example can be stated as:

$$\nabla^2 \nabla^2 w_b = 0. \tag{22.11}$$

Boundary conditions:

1) On the edge $x = 0$:

(a) If the edge $x = 0$ is of the clamped type I, then:

$$[w]_{x=0,\, z=0} = 0 \quad \text{and} \quad \left[\frac{\partial w}{\partial x}\right]_{x=0,\, z=0} = 0. \tag{22.12}$$

(b) If the edge $x = 0$ is of the clamped type II, then:

$$[w]_{x=0,\, z=0} = 0 \quad \text{and} \quad \left[\frac{\partial u}{\partial z}\right]_{x=0,\, z=0} = 0. \tag{22.13}$$

2) On the edge $x = a$:

$$[M_x]_{x=a} = 0 \quad \text{and} \quad [V_x]_{x=a} = P. \tag{22.14}$$

Procedure and solutions: The plate dimension is infinite along the y-direction. There is no variation along the y-direction in respect of boundary conditions and loading. Therefore, the problem pertains to the domain of cylindrical bending of plates (as the problem satisfies the requirements mentioned in [1, p. 4]). As a result, using Eqs. (7.1)–(7.3), the displacements u, v, w for the problem under consideration are given by:

$$u = -z\frac{\partial w_b}{\partial x} + h^3\phi_2\frac{\partial^3 w_b}{\partial x^3}, \tag{22.15}$$

$$v = 0, \tag{22.16}$$

$$w = w_b + h^2\phi_6\frac{\partial^2 w_b}{\partial x^2}. \tag{22.17}$$

Also, the solution of Eq. (22.11) can be assumed to be of the type:

$$w_b = C_1 + C_2\left(\frac{x}{a}\right) + C_3\left(\frac{x}{a}\right)^2 + C_4\left(\frac{x}{a}\right)^3, \tag{22.18}$$

where constants C_1, C_2, C_3, C_4 have dimensions of displacement.

Solution of the semi-infinite cantilever plate by the Improved 3D-RPT, when the edge $x = 0$ is of the clamped type I.

Using Eqs. (22.15) through (22.18), and conditions (22.12), (22.14), one gets:

$$w = \frac{Pa^3}{D}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3 + \frac{\mu}{2(1-\mu)}\left(\frac{h}{a}\right)^2\left(\frac{z}{h}\right)^2\left(1 - \frac{x}{a}\right)\right]. \tag{22.19}$$

Using Eq. (22.19), the mid-surface displacement $[w]_{z=0}$ of the plate is given by:

$$[w]_{z=0} = \frac{Pa^3}{D}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3\right]. \tag{22.20}$$

Using Eq. (22.20), the mid-surface displacement $[w]_{x=a,\, z=0}$ at the edge $x = a$ is given by:

$$[w]_{x=a,\, z=0} = \frac{Pa^3}{3D}. \tag{22.21}$$

Solution of the semi-infinite cantilever plate by the Improved 3D-RPT, when the edge $x = 0$ is of the clamped type II.

Using Eqs. (22.15) through (22.18), and conditions (22.13), (22.14), one gets:

$$w = \frac{Pa^3}{D}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3 + \frac{\mu}{2(1-\mu)}\left(\frac{h}{a}\right)^2\left(\frac{z}{h}\right)^2\left(1 - \frac{x}{a}\right) + \frac{1}{4(1-\mu)}\left(\frac{h}{a}\right)^2\frac{x}{a}\right]. \tag{22.22}$$

Using Eq. (22.22), the mid-surface displacement $[w]_{z=0}$ of the plate is given by:

$$[w]_{z=0} = \frac{Pa^3}{D}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3 + \frac{1}{4(1-\mu)}\left(\frac{h}{a}\right)^2\frac{x}{a}\right]. \tag{22.23}$$

Using Eq. (22.23), the mid-surface displacement $[w]_{x=a,\, z=0}$ at the edge $x = a$ is given by:

$$[w]_{x=a,\, z=0} = \frac{Pa^3}{3D}\left[1 + \frac{3}{4(1-\mu)}\left(\frac{h}{a}\right)^2\right]. \tag{22.24}$$

If a typical value of Poison's ratio μ is taken as $\mu = 0.3$, then using Eq. (22.24), the mid-surface displacement $[w]_{x=a,\, z=0}$ at the edge $x = a$ is given by:

$$[w]_{x=a,\, z=0} = \frac{Pa^3}{3D}\left[1 + 1.071\left(\frac{h}{a}\right)^2\right]. \tag{22.25}$$

22.5. Comments on results of illustrative example of the semi-infinite cantilever plate

To the best of the knowledge of the author, the two types of clamped conditions for plates were first mentioned in [10], and there are no references available wherein both these conditions have been applied together to solve plate problems.

Therefore, in order to interprete the results, it would be interesting to find out:

1) How the results obtained using the Improved 3D-RPT would compare with those obtained using the classical plate theory (CPT).
2) Whether there is any similarity about clamped conditions in the case of plates (using the Improved 3D-RPT) and in the case of beams (using the theory of elasticity approach).

22.5.1. Comparison of results obtained using the Improved 3D-RPT with those obtained using the CPT

Results obtained by the CPT:

If the CPT is used, it is easy to note that:

1) $u = -z\dfrac{w_b}{\partial x}$, $v = 0$, $w = w_b$.
   Therefore, in the CPT, both the clamped type I and clamped type II conditions would not differ from each other, and the conditions at the edge $x = 0$ would be given by Eq. (22.12).
2) The conditions at the edge $x = a$ would be given by Eq. (22.14).
3) The Eq. (22.11) would be the governing equation (as $w = w_b$).

Using the preceding information, and noting that the problem pertains to the domain of cylindrical plate bending, it is easy to show that the solution using the CPT is given by:

$$[w]_{z=0} = \frac{Pa^3}{D}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3\right] \quad \text{using the CPT.} \tag{22.26}$$

Using Eq. (22.26), the mid-surface displacement $[w]_{x=a,\, z=0}$ at the edge $x = a$ is given by:

$$[w]_{x=a,\, z=0} = \frac{Pa^3}{3D} \quad \text{using the CPT.} \tag{22.27}$$

Comparison of solutions by the Improved 3D-RPT and the CPT, when condition at the edge $x = 0$ is of the clamped type I:

From Eqs. (22.20), (22.26), it is seen that:

1) The mid-surface displacement $[w]_{z=0}$ in case of the Improved 3D-RPT, and that in case of the CPT are identical.
2) Therefore, contrary to the expectation, use of the Improved 3D-RPT shows that there is no effect of transverse stresses on the mid-surface displacement $[w]_{z=0}$.

Comparison of solutions by the Improved 3D-RPT and the CPT, when condition at the edge $x = 0$ is of the clamped type II:

From Eqs. (22.23), (22.26), it is seen that:

1) The mid-surface displacement $[w]_{z=0}$ in case of the Improved 3D-RPT includes an entity $\left(\frac{h}{a}\right)^2$, and as the entity becomes smaller and smaller (i.e., as the plate becomes thinner and thinner), the mid-surface displacement given by the Improved 3D-RPT tends towards the mid-surface displacement given by the CPT.
2) The results obtained by using the clamped type II condition differ significantly from those obtained by using the clamped type I condition.

It would be interesting to find out whether there is any similarity about clamped conditions in the case of plates (using the Improved 3D-RPT) and in the case of beams (using the theory of elasticity approach).

22.5.2. Striking similarities about clamped conditions in case of plates using the Improved 3D-RPT and in case of beams using the theory of elasticity approach

It may be noted that even though plate and beam problems are of different categories, there is a good deal of similarities in cylindrical plate bending problems and beam problems.

Fortunately, a cantilever beam problem has been solved (see [23, pp. 41–46]), using the theory of elasticity approach, wherein two types of clamped conditions have been taken into account. It would be worthwhile to look into the beam problem just mentioned. In [23], the coordinate system and notations used are different from those used in the present paper. But the beam problem and the results would be described here using the coordinate system and notations suitable for discussion in the present paper. The beam occupies the region:

$$0 \le x \le a, \quad -0.5 \le y \le 0.5, \quad -h/2 \le z \le h/2. \tag{22.28}$$

The material properties are the same as those of the plate considered here in this paper. The moment of inertia of the beam cross-section is I.

Solution of cantilever beam by the theory of elasticity approach, when condition at the end $x = 0$ is of the clamped type I:

The mid-surface deflection $[w]_{z=0}$ of the beam is given by:

$$[w]_{z=0} = \frac{Pa^3}{EI}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3\right]. \tag{22.29}$$

Using Eq. (22.29), the mid-surface displacement $[w]_{x=a,\,z=0}$ at the beam end $x = a$ is given by:

$$[w]_{x=a,\,z=0} = \frac{Pa^3}{3EI}. \tag{22.30}$$

Solution of cantilever beam by the theory of elasticity approach, when condition at the end $x = 0$ is of the clamped type II:

The mid-surface deflection $[w]_{z=0}$ of the beam is given by:

$$[w]_{z=0} = \frac{Pa^3}{EI}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1}{6}\left(\frac{x}{a}\right)^3 + \frac{(1+\mu)}{4}\left(\frac{h}{a}\right)^2\frac{x}{a}\right]. \tag{22.31}$$

Using Eq. (22.32), the mid-surface displacement $[\ w\ ]_{x=a,\ z=0}$ at the beam end $x = a$ is given by:

$$[w]_{x=a,\,z=0} = \frac{Pa^3}{3EI}\left[1 + \frac{3(1+\mu)}{4}\left(\frac{h}{a}\right)^2\right]. \tag{22.32}$$

If a typical value of Poison's ratio μ is taken as $\mu = 0.3$, then using Eq. (22.32), the mid-surface displacement $[w]_{x=a,\,z=0}$ at the beam end $x = a$ is given by:

$$[w]_{x=a,\,z=0} = \frac{Pa^3}{3EI}\left[1 + 0.975\left(\frac{h}{a}\right)^2\right]. \tag{22.33}$$

Striking similarity observed about clamped conditions in case of plates using the Improved 3D-RPT and in the case of the beams using the theory of elasticity approach

1) When the plate edge $x = 0$ and the beam end $x = 0$ both have the clamped type I conditions:

   (a) Mid-surface displacement $[w]_{z=0}$ in the case of plate given by Eq. (22.20) using the Improved 3D-RPT and in the case of beam given by Eq. (22.29) using the theory of elasticity approach are almost identical except that plate rigidity D in the case of plate gets replaced by beam rigidity EI in the case of the beam.

(b) Similarly, the mid-surface displacement $[w]_{x=a,\, z=0}$ in the case of plate given by Eq. (22.21) using the Improved 3D-RPT and in the case of the beam given by Eq. (22.30) using the theory of elasticity approach are almost identical except that plate rigidity D in the case of plate gets replaced by beam rigidity EI in the case of the beam.

2) When plate edge $x = 0$ and beam end $x = 0$ both have clamped type II conditions:

(a) The mid-surface displacement $[w]_{z=0}$ in the case of the plate given by Eq. (22.23) using the Improved 3D-RPT and in the case of the beam given by Eq. (22.31) using the theory of elasticity approach have strong similarity except for the following:

i) Plate rigidity D in the case of plate gets replaced by beam rigidity EI in the case of the beam.
ii) The last term in square brackets of Eq. (22.23) differs slightly from the last term in square brackets of Eq. (22.31).
iii) When Poison's ratio $\mu = 0.3$, the numbers in the results presented for $[w]_{x=a,\, z=0}$ in Eq. (22.25) and in Eq. (22.33) are nearly matching.

It can be observed that there are striking similarities of clamped conditions in the case of plates using the Improved 3D-RPT and in the case of beams using the theory of elasticity approach (even though the problems belong to different categories).

Therefore, there are reasons to believe that the results obtained using the Improved 3D-RPT for the semi-infinite cantilever plate are accurate in each case when the clamped edge can have the clamped type I condition or the clamped type II condition.

22.5.3. About the question regarding which clamped condition should be used
Taking a cue from the remarks in [23, p. 46] on the clamped conditions in respect of beams, the following can be suggested:

1) Both the clamped type I and clamped type II conditions are difficult to realize in practice.
2) The clamped type II condition would give better estimate of the effects of transverse stresses on transverse displacement.

The clamped type II condition (for which results are not available in plate related literature) requires further study.

22.6. Important observations on the results from the illustrative examples

Following are overall broad observations on results of the illustrative examples (dealing with flexure, vibrations, stability of plates):

1) Taking into account the information about accuracy of the CPT mentioned in Section 21 one notes that:

   (a) Even though the stresses (including transverse stresses) obtained by the CPT are reasonably accurate even for thick plates (and Table 2 confirms this for the non-dimensional flexural stress $\widehat{\sigma_x}$), most importantly, it is relevant to note here the observations of Section 20. In the CPT, stresses τ_{zx}, τ_{yz}, σ_z can only be obtained by a tedious indirect manner using the equilibrium equations; whereas, in the Improved 3D-RPT, and in the exact theories, all the stresses can be obtained directly using strain-displacement relations and constitutive relations. The results for σ_x obtained by the Improved 3D-RPT are fairly accurate.

   (b) It can be seen from Table 1 that for a thick plate (having $h/a = 0.14$) the result for $\widehat{w}$ obtained by the CPT has a large error of -8.24%. Whereas, the corresponding error for the Improved 3D-RPT is only (-0.15%). Therefore, the Improved 3D-RPT results are fairly accurate for displacements.

2) From Tables 1–4 it can be noted that the results obtained by the Improved 3D-RPT in the case of flexure, vibrations, stability are, in general, closer to – and sometimes marginally superior to – those obtained by theories of Reissner, Mindlin, Reddy. The Improved 3D-RPT is easy to use.
3) From Sections 22.5.2 and 22.5.3 it can be believed that the results obtained for both the clamped conditions are accurate.

23. Areas for further studies

1) The study is limited to the isotropic and homogeneous material.

   Satisfaction of all the equations, conditions involved in respect of general plate problems of other materials can be a challenging task. With some efforts, it should be possible to extend the proposed plate theories to other problems related to general plate problems of other materials.
2) The clamped type II condition (for which results are not available in plate related literature) requires further study.

24. Conclusions

In this paper, "Improved three-dimensional refined plate theory" (Improved 3D-RPT) has been introduced.

1) The Improved 3D-RPT is a three-dimensional displacement based theory, which can be used for plate flexure, plate vibrations and also when the plate is under combined action of lateral loads and in-plane forces.
2) (a) The theory takes into account all the stresses appropriately:
      i) Bending stresses, in-plane shear stress: linear, as well as, nonlinear components are included.
      ii) Transverse shear stresses: realistic parabolic variation across the thickness is taken into account satisfying zero transverse stresses at the surfaces of the plate.
      ii) Transverse normal stress: realistic cubic variation across the thickness is taken into account satisfying transverse normal stress conditions at plate surfaces.
   (b) The theory satisfies all strain-displacement relations.
   (c) The theory tries to satisfy, as accurately as possible, all constitutive relations.
   (d) The theory satisfies gross equilibrium equations.
3) The theory uses the concept of targeted displacements, (which are components of displacements) which contribute only towards specific moments, shear forces, and stresses.

   The targeted displacements are used to reduce the number of unknown variables involved.
4) The number of unknown variables involved are the same as those associated with thin plates, *viz.* only one in the case of flexure and vibrations; and three in the case of stability.
5) Boundary conditions:

   (a) Previously, in a certain class of axiomatic, higher-order shear deformation theories, static inconsistencies were observed while modelling the flexural behaviour of beams, plates and shells with clamped boundary conditions.

      The clamped boundary conditions prescribed in this paper do not suffer from such inconsistencies. The clamped type II condition (for which references are not available in literature) requires further study.
   (b) In this paper two types of clamped conditions (i.e., clamped type I and clamped type II) were introduced. It was observed that there are striking

similarities about the clamped conditions in the case of plates using the Improved 3D-RPT and in the case of beams using the theory of elasticity approach.

6) There are some noteworthy significant similarities of the Improved 3D-RPT with the CPT and other theories, and the knowledge gained over in dealing with these theories would prove to be useful.

   (a) *Plate flexure*: The governing equation of the Improved 3D-RPT has striking similarity with that of the CPT save for the appearance of w_b in the governing equation of the Improved 3D-RPT, whearas w appears in the context of the CPT. Except for the clamped type II boundary conditions, all other boundary conditions of the Improved 3D-RPT have good amount of similarity with those of the CPT.
   (b) *Plate vibrations*: There is significant similarity of the governing equation of the Improved 3D-RPT with the following earlier theories:
      i) Mindlin's theory,
      ii) Levinson's theory, and
      iii) "A single variable refined theory for free vibrations of a plate using inertia related terms in displacements" (SVRPT).
   (c) *Plate under combined action of lateral loads and specified in-plane forces*: The governing equation for the plate under combined action of lateral loads and in-plane forces in the Improved 3D-RPT has noteworthy significant similarity with the corresponding equation in respect of the thin plate obtained by Saint-Venant.

7) Results of the illustrative examples obtained by the Improved 3D-RPT bring out the efficacy of the theory. The results obtained are, in general, closer to those obtained by the theories of Reissner, Mindlin, Reddy. This is remarkable in view of the simplicity of the Improved 3D-RPT in respect of governing equations, boundary conditions.

In short, the "Improved three-dimensional refined plate theory" (Improved 3D-RPT) presented here is simple, easy to use, accurate, three-dimensional plate theory.

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Received July 26, 2024; revised version April 4, 2025.
Published online June 26, 2025.